Standard IX

MATHEMATICS

PART-I

Government of Kerala
Department of Education

State Council of Educational Research and Training (SCERT)
2016

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Mathematics starts as the study of measurements and the relations between measurements. The conceptual level of mathematics develops when measurements are seen as pure numbers and physical objects as geometrical entities. Number relation evolves into algebraic equations. To interpret new situations mathematically, new numbers and novel techniques are needed. Cause and effect relations between facts develop into logical connections of ideas. Thus Mathematics grows. Welcome to the next step!

With love and regards

Dr. P. A. Fathima
Director, SCERT

TEXTBOOK DEVELOPMENT

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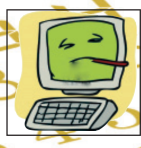

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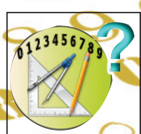
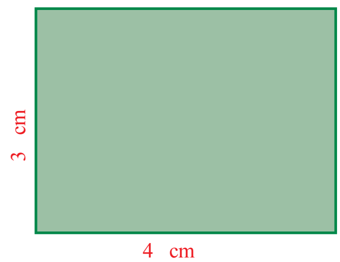


For Discussion

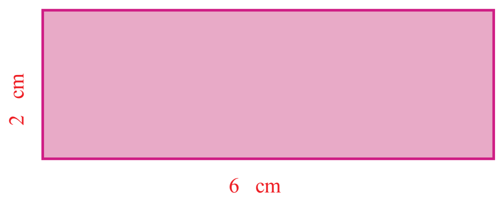
1 Area

We have to draw a rectangle of area 12 square centimetres. How?

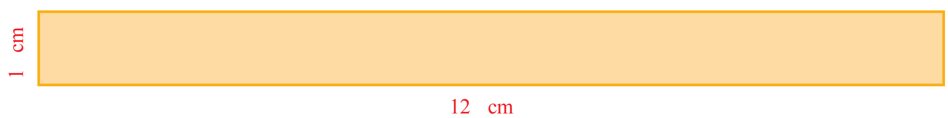
It can be like this:



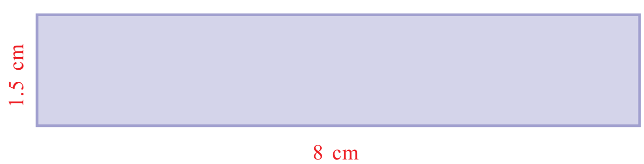
Or like this:



And there are so many other ways, right?

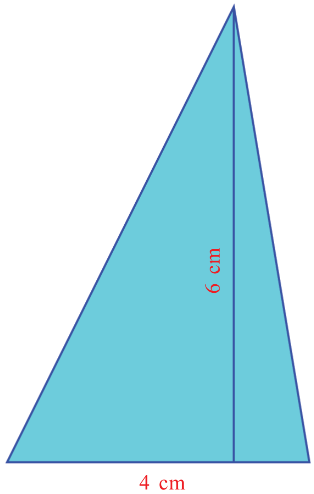


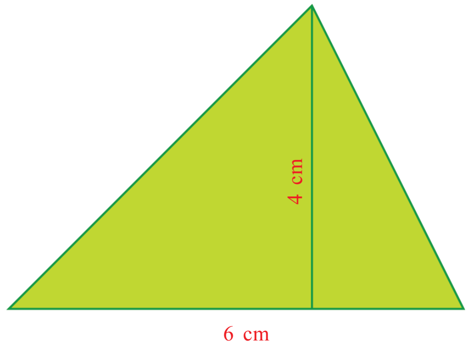
Suppose we also want one side to be 8 centimetres long. There is only one such rectangle, isn't it?



What if we want a triangle of area 12 square centimetres?

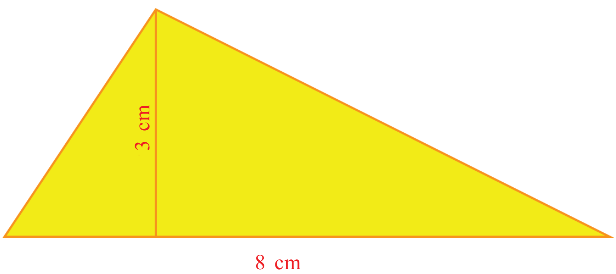
Again it can be done in many ways:

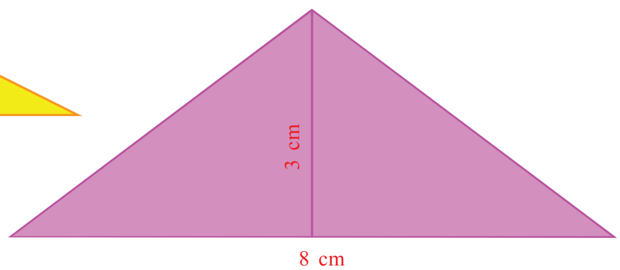


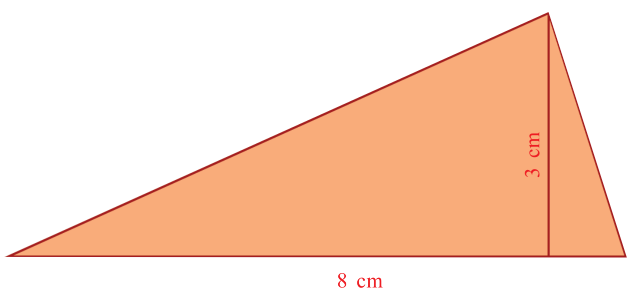


And if we want one side to be 8 centimetres?

Then also there are several:

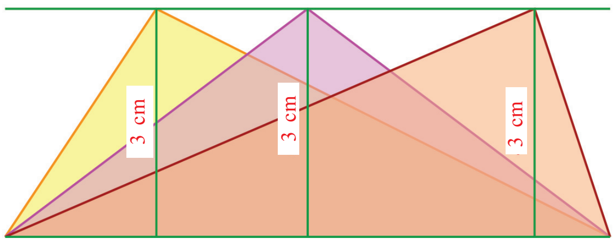






Draw a line 8 centimetres long and mark a point at a height 3 above it (we can use **Grid** for this). Draw a line through this point, parallel to the first. Mark a point on this line and draw a triangle with this point and the end points of the first line as vertices. Use **Area** to mark its area. Shift the top vertex along the parallel line. Do you see any change in area?

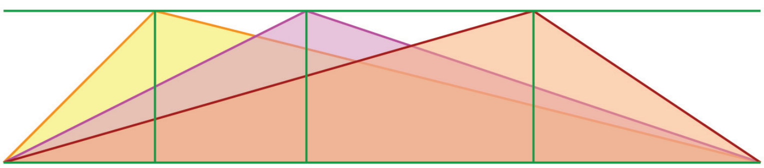
In all these the uppersides are different. Since the base and the height are the same, so is the area.



The top vertices of all these triangles lie 3 centimetres from the base. In other words, they are all on the line parallel to the base, at a distance 3 centimetres from it.

The top vertex of all triangles with this base and area must lie on this line. On the other hand, if we join any point on this line with the end points of the bottom line, we get a triangle of this base and area.

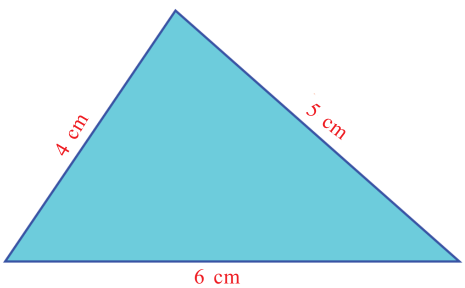
This is true whatever be the base and area, right?



All triangles with the same base and area have their third vertices on a line parallel to the base; conversely, all triangles with the same base and the third vertex on a line parallel to the base, have the same area.

Let's see some ways of putting this to use.

Draw a triangle of sides 4 centimetres, 5 centimetres and 6 centimetres.

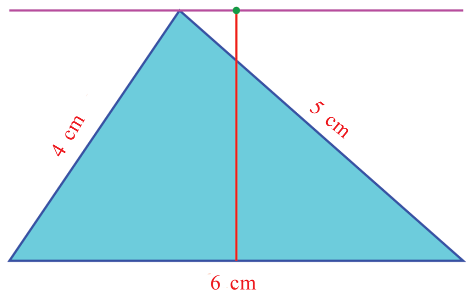


Now can we draw an isosceles triangle with the same base and area?

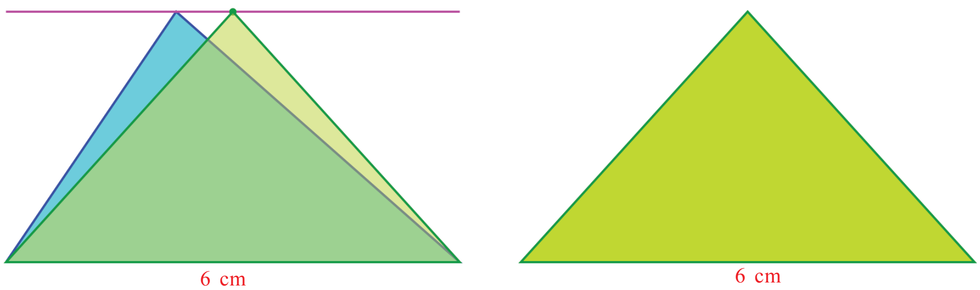
Since we don't want to change the base, the only question is where to mark the top vertex. To have the same area, it must be on the line through the top vertex of the triangle we have drawn and parallel to the base.

And we have seen in Class 8 that, the top vertex of any isosceles triangle is on the perpendicular bisector of the base.

So the third vertex we seek is the point where the perpendicular bisector of the base of the drawn triangle and the line through its top vertex parallel to the base intersect.

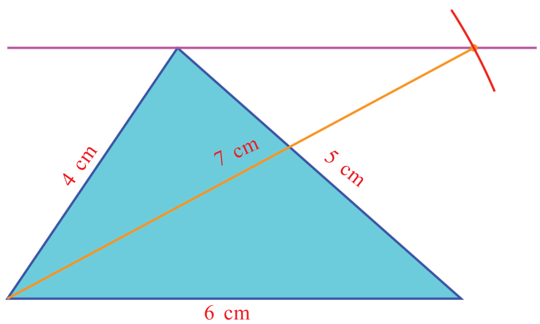


Now we can draw the required triangle:

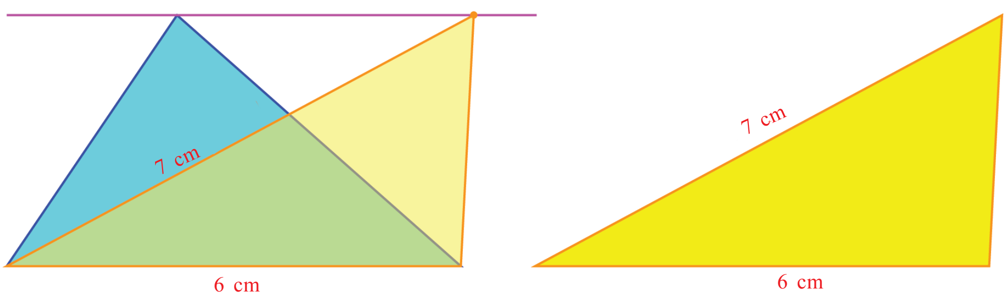


Again can you draw another triangle of the same base and area with the left side 7 centimetres?

All we need to do is to draw an arc of radius 7 centimetres, centred at the left vertex and intersecting the parallel line on top, right?



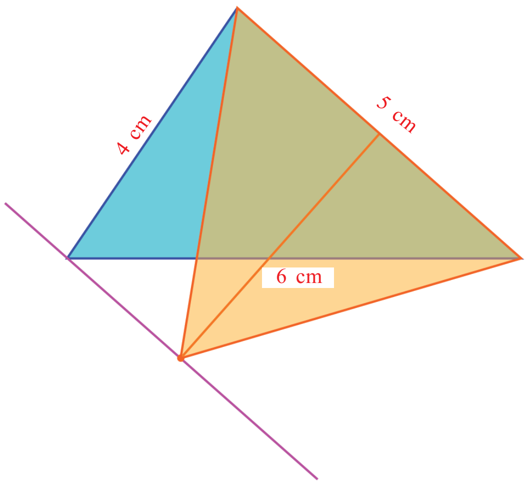
Thus we get a triangle like this:

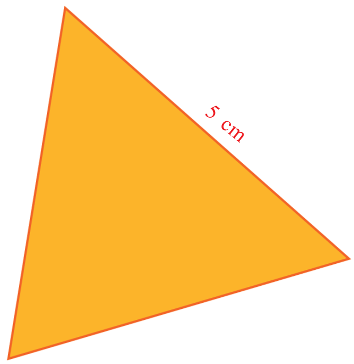


Can we draw an isosceles triangle of the same area with base 5 centimetres?

We can redraw our first picture with the 5 centimetre side at the bottom and proceed as before.

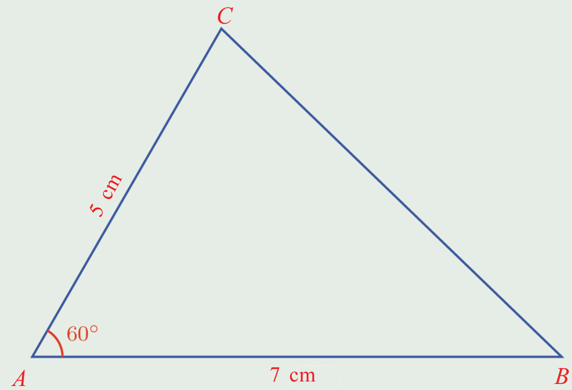
If we don't mind a tilted triangle, then we can do this using the first drawn triangle itself, by drawing a line through the left vertex parallel to the right side.





(1) Draw a triangle of sides 3, 4 and 6 centimetres. Draw three different right triangles of the same area.

(2) Draw the triangle shown below in your notebook:



Draw triangles *ABP*, *BCQ* and *CAR* of the same area with measurements given below:

i) $\angle BAP = 90^\circ$

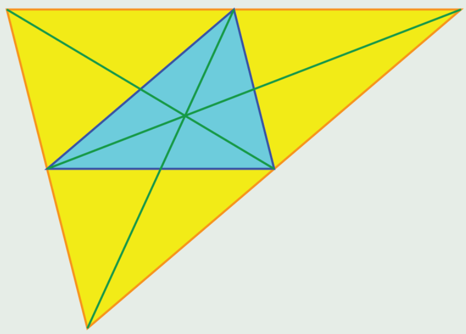
ii) $\angle BCQ = 60^\circ$

iii) $\angle ACR = 30^\circ$

(3) Draw a circle and a triangle with one vertex at the centre of the circle and the other two on the circle. Draw another triangle of the same area with all three vertices on the circle.

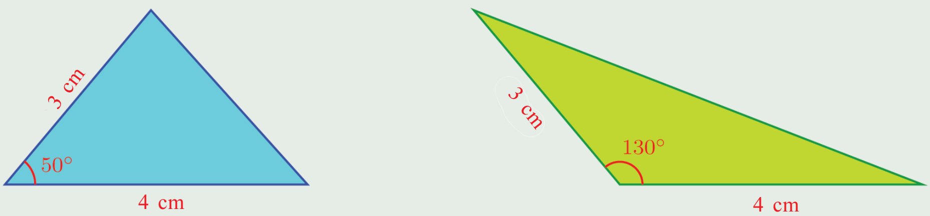
(4) How many different (non - congruent) triangles can you draw with two sides 8 and 6 centimetres and area 12 square centimetres? What if the area is to be 24 square centimetres?

(5) In the picture below, the lines parallel to each side of the blue triangle through the opposite vertex are drawn to make the big triangle.



How many triangles in the picture have the same area as that of the blue triangle?

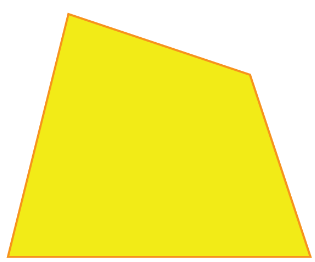
(6) Prove that the two triangles shown below have the same area:



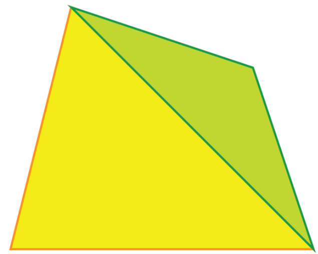
How many different triangles of the same area can be drawn without changing the lengths of two sides?

Quadrilateral and triangle

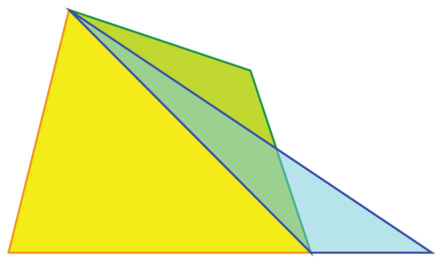
How do we calculate the area of an ordinary quadrilateral without any special properties?



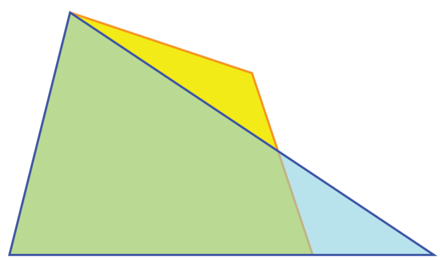
Draw a diagonal to divide it into two triangles and compute the area of each, right?



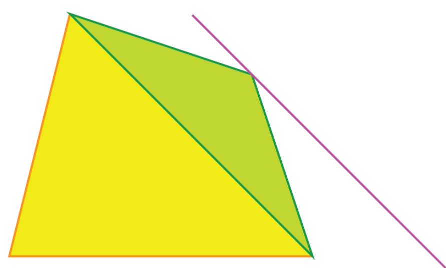
There is another way. Suppose we can bring down the top right vertex of the green triangle to the base of the quadrilateral without altering the area:



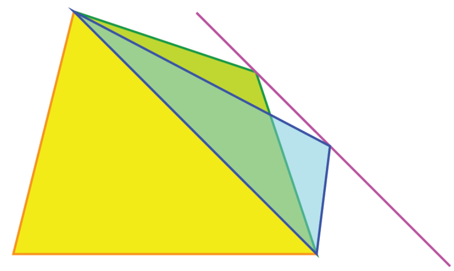
Then the area of the quadrilateral would be the sum of the yellow and blue triangles. And these two triangles together form a big triangle. Thus we can convert the area of the quadrilateral to the area of a single triangle.



Now we see how this wish can be realised. To shift a vertex of the green triangle without altering base and area, we need only draw a line through this vertex, parallel to the opposite side:

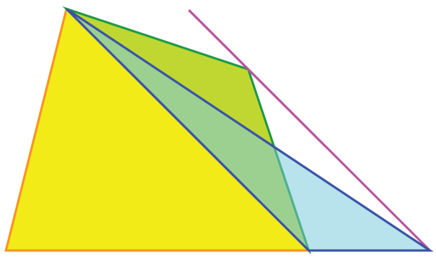


However much we slide the top right vertex of the green triangle along this line, its area won't change; so the area of the new quadrilateral thus got also does not change.

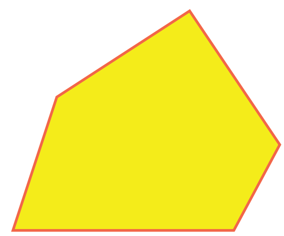


What if we slide the vertex all the way down to the point where the parallel line meets the base of the quadrilateral extended?

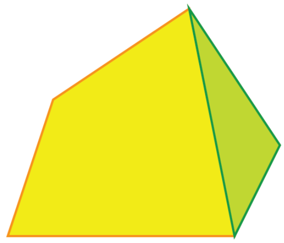
We get a triangle of the same area as the quadrilateral, right?



Repeating this trick again and again we can make a triangle of the same area as any polygon. For example, look at this pentagon:



By joining two alternate vertices we can split it into a quadrilateral and a triangle:

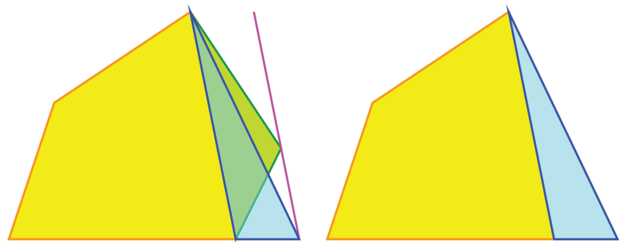


Cutting and rearranging

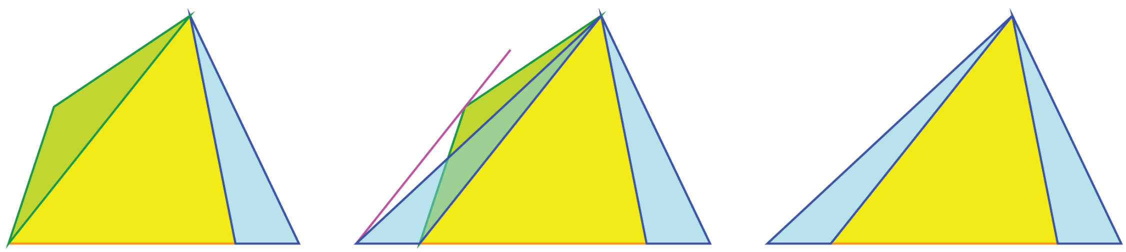
If we cut a paper figure and rearrange the pieces to another shape, the area is not altered. A geometric method of changing a figure into another may not always give a method to cut and rearrange. For example, the method of drawing a triangle of the same area as that of a quadrilateral cannot be used to cut a paper quadrilateral and rearrange the pieces to form a triangle. Several websites which give such dissection methods are given in the webpage www.cs.purdue.edu/homes/gnf/book/webdiss.html

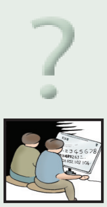
Draw a quadrilateral, a pentagon and a hexagon in GeoGebra and draw triangles of the same areas.

Now sliding down the top right corner of the green triangle parallel to the opposite side to the base of the pentagon; we get a quadrilateral of the same area:

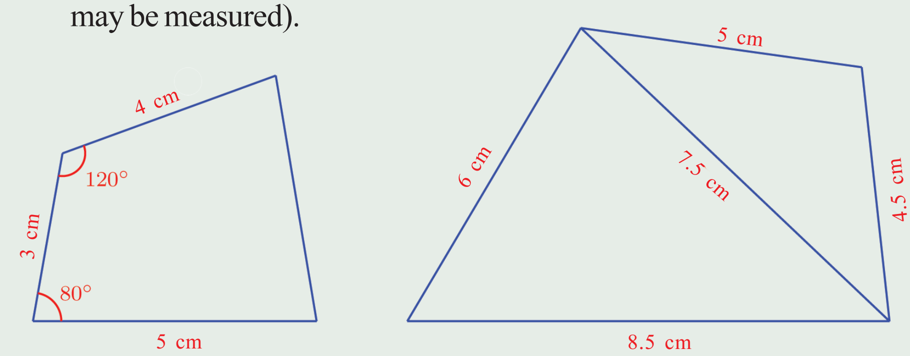


Sliding down the top left vertex of the quadrilateral also like this, we get a triangle of the same area:





(1) Draw the two quadrilaterals shown below, in your note book. Draw triangles of the same area and calculate the areas (The lengths needed may be measured).

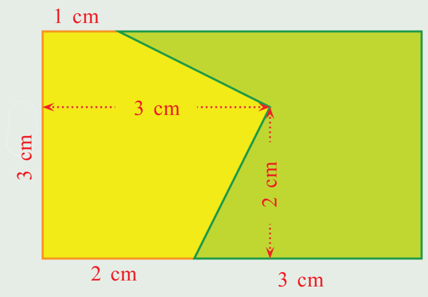


(2) Draw a rhombus of sides 6 centimetres and one angle 60°; then draw a right triangle of the same area.

(3) Draw a regular pentagon and then a triangle of the same area. Calculate the area.

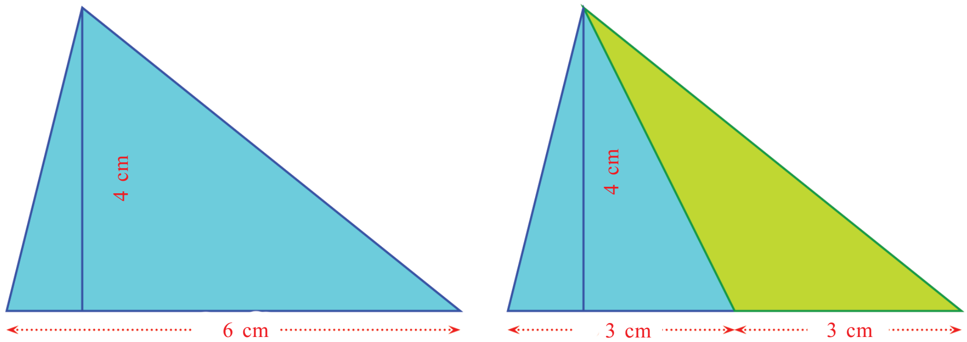
(4) The picture shows a rectangle divided into two parts.

Instead of the broken line separating these parts, draw a straight line to divide the rectangle into two other parts of the same area. Calculate the areas of these parts.



Triangle division

See these pictures:



The line joining one vertex of the triangle to the midpoint of the opposite side divides it into two triangles.

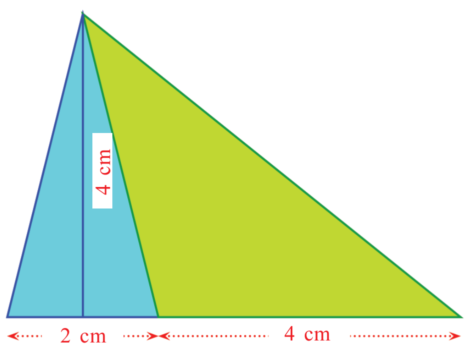
What are the areas of these parts?

Both have bases of 3 centimetres.

And the heights? It's 4 centimetres for both.

So they have the same area, 6 centimetres.

Now suppose the top vertex is joined to some other point of the bottom side. For example, see this picture:



Now the smaller triangle is of area 4 square centimetres and the larger one is of area 8 square centimetres.

Thus the area of the larger triangle is twice the area of the smaller triangle. The bottom side is also divided in the same manner, right? The longer part is twice the shorter part.

We can put it in terms of ratios.

The bottom side is divided in the ratio 1 : 2; and the area of the triangle is also divided in the same ratio.

Is this true for other ways in which the line from the top vertex divides the bottom side?

What if this ratio is 2 : 3?

The lengths would be as below:

Length of the shorter side $6 \times \frac{2}{5}$ centimetres

Length of the longer side $6 \times \frac{3}{5}$ centimetres

And the areas?

Area of the smaller part $6 \times \frac{2}{5} \times 2 = 12 \times \frac{2}{5}$ square centimetres

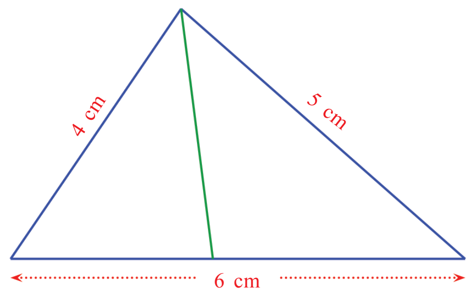
Area of the larger part $6 \times \frac{3}{5} \times 2 = 12 \times \frac{3}{5}$ square centimetres

Thus the line from the top divides the area of the whole triangle in the ratio 2 : 3.

We can see in this way that whatever the ratio of lengths, the same is the ratio of areas. Even when we change the dimensions of the triangle, this fact does not change.

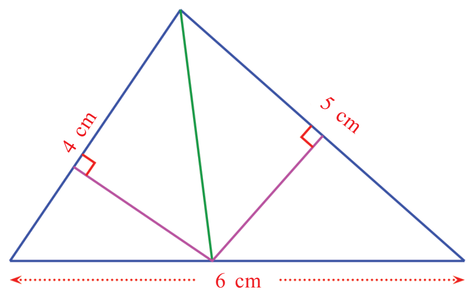
A line from the vertex of a triangle divides the length of the opposite side and the area of the triangle in the same ratio.

We saw that the bisector of a side of a triangle, through the opposite vertex bisects the triangle also. So a natural question is, in what ratio does the bisector of an angle at one vertex divide the opposite side (and hence the triangle)?



The picture shows the angle bisector of the top vertex of the triangle.

We have to compute the ratio in which it divides the opposite side.



Here we know the lengths of one side of both the triangular parts. So let's compute the area also in terms of these. For that, we draw the perpendiculars from the opposite vertex. It is the same point for both triangles.

Don't these perpendiculars seem to have the same length?

Let's check. The upper right triangles on either side in the picture have the same hypotenuse. Since it is the angle bisector of the top angle of the large triangle, the top angle of these smaller triangles are equal. Since these triangles are right, the angle at the other end of the hypotenuse is also equal. So the perpendicular sides of these triangles are also equal. Thus the perpendiculars we have drawn are of the same length.

So, the areas of these triangles are 4 and 5 multiplied by half this length. That is, they are in the ratio 4 : 5.

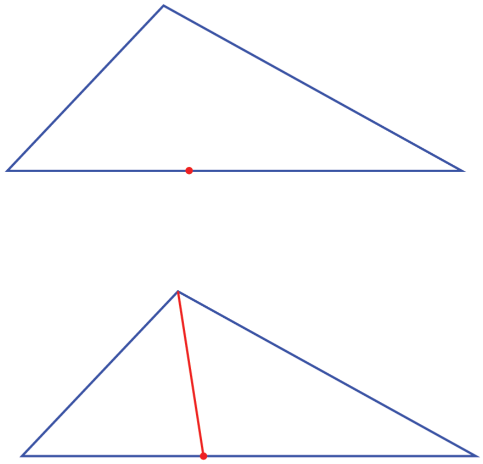
As seen earlier, the angle bisector divides the opposite side also in the same ratio.

And this is true whatever be the lengths of the sides of the triangle.

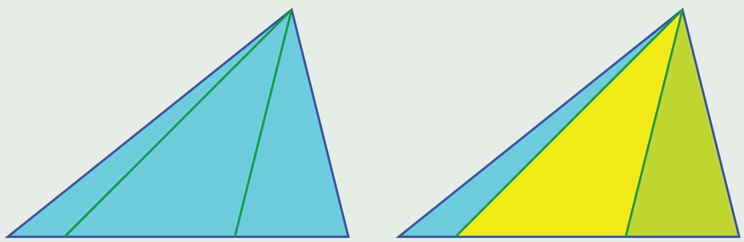
In any triangle, the bisector of an angle divides the opposite side in the ratio of the sides of the angle.

This can be put in another way:

In the picture, the point marked on the bottom side divides that side in the ratio of the other two sides. As seen just now, the bisector of the top angle passes through this point. That is, the line joining the top vertex and this point is the bisector of the top angle.

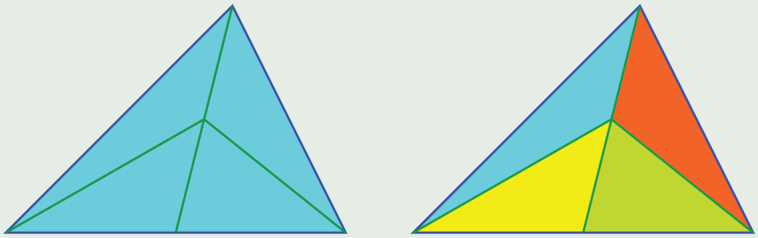


(1) In the picture below, two lines are drawn from the top vertex of a triangle to the bottom side:



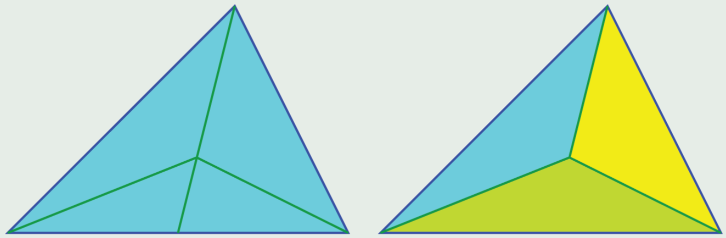
Prove that the ratio in which these lines divide the length of the bottom side is equal to the ratio of the area of the three smaller triangles in the picture.

(2) In the picture below, the top vertex of a triangle is joined to the mid point of the bottom side of the triangle and then the mid point of this line is joined to the other two vertices.



Prove that the areas of all four triangles obtained thus are equal to a fourth of the area of the whole triangle.

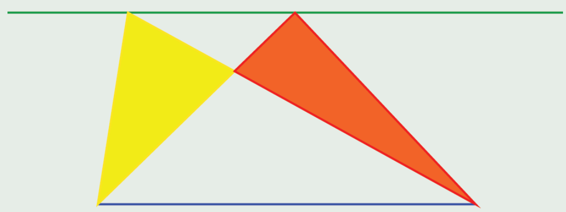
(3) In the picture below, the top vertex of a triangle is joined to the mid point of the opposite side and then the point dividing this line in the ratio 2 : 1 is joined to the other two vertices:



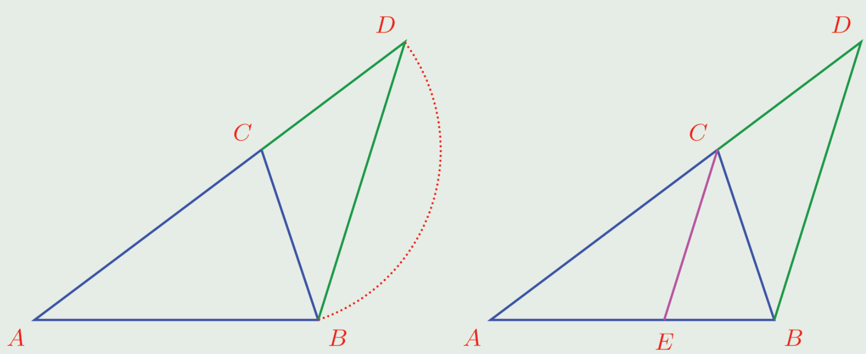
Prove that the areas of all three triangles in the picture on the right are equal to a third of the area of the whole triangle.

(4) Prove that the lengths of the perpendiculars from any point on the bisector of an angle to the sides are equal.

(5) In this picture the horizontal lines at the top and bottom are parallel. Prove that the yellow and red triangles are of the same area.



(6) In the picture below, the side AC of the triangle ABC is extended to D, by adding the length of the side CB. Then the line through C parallel to DB is drawn to meet AB at E.

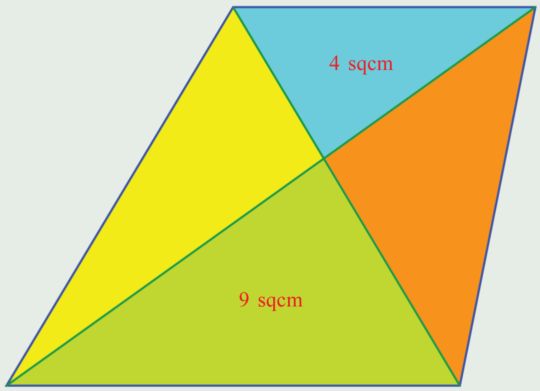


i) Prove that CE bisects $\angle C$.

ii) Describe how this can be used to divide an 8 centimetres long line in the ratio 4 : 5.

iii) Can we use it to divide an 8 centimetres long line in the ratio 3 : 4? How?

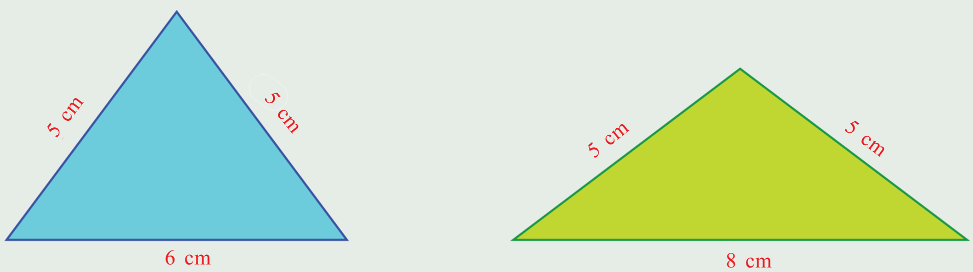
(7) The picture below shows a trapezium divided into four parts by the diagonals.



The area of the blue triangle is 4 square centimetres and the area of the green triangle is 9 square centimetres. What is the total area of the trapezium?

(8) Draw a square of sides 5 centimetres. Draw an isosceles triangle of the same area.

(9) Prove that the two triangles below have the same area:



Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Explaining the methods of drawing triangles of the same area as a given triangle.			
• Justifying the methods to transform a polygon into other polygons of the same area.			
• Converting polygons into triangles and computing the areas.			
• Finding out methods to divide the area of a triangle in a specified ratio.			
• Solving geometric problems related to areas.			

2 Fractions

Equal fractions

We have seen that a single fraction has many forms. For, example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$$

Here, $\frac{2}{4}, \frac{3}{6}, \dots$ are all different forms of $\frac{1}{2}$. We can also say that all these fractions are equal to $\frac{1}{2}$.

Similarly, we have

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \dots$$

So that $\frac{6}{10}, \frac{9}{15}, \dots$ are all different forms of $\frac{3}{5}$; or fractions equal to $\frac{3}{5}$.

When we multiply the numerator and denominator of a fraction by the same natural number, we get different forms of it; that is, fractions equal to it.

We can put this in algebra:

For any fraction $\frac{a}{b}$ and any natural number n,

$$\frac{an}{bn} = \frac{a}{b}$$

Now let's look at some patterns. For example,

$$\frac{1^2 + 1}{1 + 1} = 1 \qquad \frac{2^2 + 2}{2 + 1} = 2 \qquad \frac{3^2 + 3}{3 + 1} = 3$$

Is this true for all natural numbers? For instance, is $\frac{138^2+138}{138+1}$ equal to 138 itself?

We can compute the square, add and divide to check it. But that is not very interesting and we won't know why it happens either.

Instead, let's rewrite the numerator:

$$138^2 + 138 = 138\,(138 + 1)$$

Now taking the denominator also, we get

$$\frac{138^2 + 138}{138 + 1} = \frac{138(138 + 1)}{138 + 1} = 138$$

This we can do to any fraction of this kind, right?

Let's write it using algebra:

$$\frac{n^2 + n}{n + 1} = \frac{n\,(n + 1)}{n + 1} = n$$

Explain each of the patterns below and write the general principle in algebra.

(1) $\frac{1^2 + 1}{1 + 1} = 1$ $\quad \frac{2^2 + 2}{2 + 2} = 1\frac{1}{2}$ $\quad \frac{3^2 + 3}{3 + 3} = 2$ $\quad \frac{4^2+4}{4+4} = 2\frac{1}{2}$

(2) $\frac{2^2-2}{2-1} = 2$ $\quad \frac{3^2-3}{3-1} = 3$ $\quad \frac{4^2-4}{4-1} = 4$ $\quad \frac{5^2-5}{5-1} = 5$

(3) $\frac{2^2-1}{2-1} = 3$ $\quad \frac{3^2-1}{3-1} = 4$ $\quad \frac{4^2-1}{4-1} = 5$ $\quad \frac{5^2-1}{5-1} = 6$

Cross multiplication

It we multiply the numerator and denominator of a fraction by the same natural number, we get another form of it. But two forms of the same fraction may not have such a relation.

For example, we have $\frac{2}{4} = \frac{3}{6}$. But we can't get 3 and 6 by multiplying 2 and 4 by the same natural number.

So how do we check whether two fractions are equal?

One way is to remove the common factors in the numerator and denominator and reduce to the lowest terms.

For example, take $\frac{42}{63}$ and $\frac{70}{105}$:

$$\frac{42}{63} = \frac{2 \times 3 \times 7}{3 \times 3 \times 7} = \frac{2}{3}$$

$$\frac{70}{105} = \frac{2 \times 5 \times 7}{3 \times 5 \times 7} = \frac{2}{3}$$

Thus we see that $\frac{42}{63}$ and $\frac{70}{105}$ are two forms of $\frac{2}{3}$.

It is not easy to factorise large numbers (even if we use a computer). So we use another method to check equality of the fractions. For example, take $\frac{119}{221}$ and $\frac{133}{247}$.

They can be put in forms with the same denominator (Remember how we did this to add two fractions in Class 5?)

$$\frac{119}{221} = \frac{119 \times 247}{221 \times 247}$$

$$\frac{133}{247} = \frac{133 \times 221}{247 \times 221}$$

The denominator of these new forms are the same. So to check their equality, we need only see whether the numerators are also equal.

We can do this quickly using a calculator.

$$119 \times 247 = 29393$$

$$133 \times 221 = 29393$$

Since the numerators and denominators are equal, so are the fractions.

$$\frac{119}{221} = \frac{133}{247}$$

We can write all this in algebra. To check whether the fractions $\frac{a}{b}$ and $\frac{p}{q}$ are equal, we first put them in forms with the same denominator.

$$\frac{a}{b} = \frac{aq}{bq} \qquad \frac{p}{q} = \frac{bp}{bq}$$

Now to check equality, we need only see if the numerators aq and bp are equal.

On the other hand, suppose that for some natural numbers a, b, p, q, we have aq and bp equal.

Then the fractions $\frac{aq}{bq}$ and $\frac{bp}{bq}$ have both their numerators and denominators equal, so that

$$\frac{aq}{bq} = \frac{bp}{bq}$$

Now removing the common factors of numerator and denominator of each, we get

$$\frac{a}{b} = \frac{p}{q}$$

For natural numbers a, b, p, q if $\frac{a}{b} = \frac{p}{q}$, then $aq = bp$;

conversely, if $aq = bp$, then $\frac{a}{b} = \frac{p}{q}$

This method of converting equality of fractions to equality of products of natural numbers is often called *cross multiplication*.

$$\frac{a}{b} \times \frac{p}{q}$$

From two forms of a fraction, we can make many, more forms. For example, $\frac{2}{4}$ is another form of $\frac{1}{2}$. See what we get by adding the numerators and denominators:

$$\frac{1+2}{2+4} = \frac{3}{6}$$

This is another form of $\frac{1}{2}$, isn't it?

Now let's do this with $\frac{1}{2}$ and $\frac{3}{6}$:

$$\frac{1+3}{2+6} = \frac{4}{8}$$

It's another form of $\frac{1}{2}$, isn't it?

We can do this with $\frac{2}{4}$ and $\frac{3}{6}$ also.

$$\frac{2+3}{4+6} = \frac{5}{10}$$

Why does this happen?

To understand the inner workings of number relations, we need algebra.

Let's start with

$$\frac{a}{b} = \frac{p}{q}$$

We want to know why $\frac{a+p}{b+q}$ is also equal to these.

Since $\frac{a}{b}$ and $\frac{p}{q}$ are equal, we have

$$aq = bp$$

Now if the fractions $\frac{a+p}{b+q}$ and $\frac{a}{b}$ are to be equal, the products $(a+p)b$ and $(b+q)\,a$ must be equal, by cross multiplication. Let's look at each:

$$(a+p)\,b = ab + pb$$
$$(b+q)\,a = ba + qa$$

ab and *ba* are the same. So the final form of each product contains *ab*. What remains in the first product is *pb*; and in the second product, *qa*. In these, *pb* is the same as *bp*, and *qa* the same as *aq*; and we have noted earlier that these are equal. So, the products are equal, right?

From a pair of equal fractions, we can make another pair. For example we have

$$\frac{5}{2} = \frac{10}{4}$$

We make new fractions using the sums and differences of the numerator and denominator, like this:

$$\frac{5+2}{5-2} = \frac{7}{3}$$

$$\frac{10+4}{10-4} = \frac{14}{6} = \frac{7}{3}$$

Let's take another pair:

$$\frac{4}{2} = \frac{16}{8}$$

An innocent question

Little brother in Class 6 asks:

"Teacher says $\frac{1}{2}$ and $\frac{2}{4}$ are equal. But taking 1 candy from 2, and 2 candies from 4 aren't the same. I get only one candy in the first."

What's your answer?

As before, we find

$$\frac{4+2}{4-2} = 3$$

$$\frac{16+8}{16-8} = \frac{24}{8} = 3$$

Starting with any pair of equal fractions and using the sums and differences of the numerator and denominator like this, can we get another pair of equal fractions?

Translated to algebra, the question is this: if $\frac{a}{b}$ and $\frac{p}{q}$ are equal, are $\frac{a+b}{a-b}$ and $\frac{p+q}{p-q}$ equal?

Taking $\frac{a}{b} = \frac{p}{q}$, we get $aq = bp$. Now to check whether $\frac{a+b}{a-b}$ and $\frac{p+q}{p-q}$ are equal using cross multiplication, we need only see whether the products $(a+b)(p-q)$ and $(a-b)(p+q)$ are equal.

$$(a+b)(p-q) = ap - aq + bp - bq$$

If we use $aq = bp$ in the expression on the right side of this equation, it is reduced to $ap - bq$. Thus

$$(a+b)(p-q) = ap - bq$$

Similarly,

$$(a-b)(p+q) = ap + aq - bp - bq$$

and the right side of this can be reduced to $ap - bq$, so that

$$(a-b)(p+q) = ap - bq$$

Thus both the numbers $(a+b)(p-q)$ and $(a-b)(p+q)$ are reduced to the same number $ap - bq$. So,

$$\frac{a+b}{a-b} = \frac{p+q}{p-q}$$

(1) Look at this method of making a pair of equal fractions from another such pair:

$$\frac{1}{3} = \frac{2}{6} \rightarrow \frac{1}{2} = \frac{3}{6}$$

$$\frac{3}{4} = \frac{9}{12} \rightarrow \frac{3}{9} = \frac{4}{12}$$

i) Check some more pairs of equal fractions. By interchanging the numerator of one with the denominator of the other, do you get equal fractions?

ii) Write this as a general principle using algebra and explain it.

(2) Look at these calculations:

$$\frac{1}{2} = \frac{2}{4} \qquad \frac{(3 \times 1) + (4 \times 2)}{(3 \times 2) + (4 \times 4)} = \frac{11}{22} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{3}{6} \qquad \frac{(3 \times 1) + (4 \times 3)}{(3 \times 2) + (4 \times 6)} = \frac{15}{30} = \frac{1}{2}$$

i) Take some more fractions equal to $\frac{1}{2}$ and form fractions by multiplying the numerators and denominators by 3 and 4 and adding. Do you get fractions equal to $\frac{1}{2}$?

ii) Take some other pairs of equal fractions and check this.

iii) In all these, instead of multiplying numerators and denominators by 3 and 4, multiply by some other numbers and add. Do you still get equal fractions?

iv) Explain why, if the fraction $\frac{p}{q}$ is equal to the fraction $\frac{a}{b}$, then for any pair of natural numbers m and n, the fractions $\frac{ma + np}{mb + nq}$ is equal to $\frac{a}{b}$.

(3) The sum of the square of a number and one, divided by the difference of 1 from the square gives $\frac{221}{220}$. What is the number?

(4) The sum of a number and its square is one and a half times their difference. What is the number?

Large and Small

Which is larger, $\frac{2}{5}$ or $\frac{3}{5}$?

How did you decide $\frac{3}{5}$ is larger?

$\frac{2}{5}$ is made up of 2 one fifths; we need 3 such parts to make $\frac{3}{5}$. So, $\frac{3}{5}$ is larger.

We can write this in short like this:

$$\frac{2}{5} < \frac{3}{5}$$

In general, of two fractions with the same denominator, the one with the larger numerator is the larger number. We can put it like this: in a fraction, if we increase the numerator alone, the fraction becomes larger. For example,

$$\frac{1}{5} < \frac{2}{5} < \frac{3}{5} < \frac{4}{5}$$

On the other hand, what if we increase the denominator alone?

For example, Let's take $\frac{3}{4}$ and $\frac{3}{5}$.

Each of 4 equal parts is larger than each of 5 equal parts:

So 3 of the first parts is larger than 3 of the second parts.

That is, $\frac{3}{5} < \frac{3}{4}$

In general, of two fractions with the same numerator, the one with the smaller denominator is the larger number. We can put it like this: In a fractions, if we decrease the denominator alone, the fraction becomes larger. For example,

$$\frac{5}{9} < \frac{5}{8} < \frac{5}{7} < \frac{5}{6}$$

So, which is larger , $\frac{3}{7}$ or $\frac{3}{5}$?

As seen now, by changing $\frac{3}{7}$ to $\frac{3}{5}$, we get a larger fraction; changing $\frac{3}{5}$ to $\frac{4}{5}$ gives a still larger fraction. That is,

$$\frac{3}{7} < \frac{3}{5} < \frac{4}{5}$$

Similarly, taking $\frac{5}{6}$ and $\frac{4}{9}$, we find

$$\frac{4}{9} < \frac{5}{9} < \frac{5}{6}$$

In general, by increasing the numerator and decreasing the denominator of a fraction, we get a larger fraction.

For example,

$$\frac{1}{10} < \frac{2}{9} < \frac{3}{7} < \frac{4}{5}$$

Now look at $\frac{1}{2}$ and $\frac{2}{3}$. Which is larger?

None of the methods seen so far help us here. (Why?)

Let's look at forms of these with the same denominators:

$$\frac{1}{2} = \frac{3}{6} \qquad \frac{2}{3} = \frac{4}{6}$$

Of these, the one with the larger numerator is the larger fraction. So, $\frac{3}{6} < \frac{4}{6}$.

Reverting to the original forms, we have $\frac{1}{2} < \frac{2}{3}$

Similarly, to find the larger of $\frac{3}{4}$ and $\frac{5}{7}$, we first put them in forms with the same denominator:

$$\frac{3}{4} = \frac{21}{28} \qquad \frac{5}{7} = \frac{20}{28}$$

We can see $\frac{5}{7} < \frac{3}{4}$, by looking at just the numerators of these forms. Let's write this method as a general principle, using algebra: To find the larger of $\frac{a}{b}$ and $\frac{p}{q}$, we first put them in forms with the same denominator:

$$\frac{a}{b} = \frac{aq}{bq} \qquad \frac{p}{q} = \frac{bp}{bq}$$

So, if $\frac{a}{b} < \frac{p}{q}$ then we have $aq < bp$.

Conversely, suppose that for some natural numbers a, b, p, q, we have $aq < bp$. This means of the fractions $\frac{aq}{bq}, \frac{bp}{bq}$, the first has the smaller numerator. Since their denominators are the same, this means that $\frac{aq}{bq} < \frac{bp}{bq}$. Removing the common factors in the numerator and denominator, we get. $\frac{a}{b} < \frac{p}{q}$.

For natural numbers a, b, p, q, if $\frac{a}{b} < \frac{p}{q}$, then $aq < bp$; conversely, if $aq < bp$, then $\frac{a}{b} < \frac{p}{q}$

Thus we can use cross multiplication to compare two fractions also.

Now let's look at some problems.

In $\frac{2}{3}$ if we add 1 to both the numerator and the denominator, we get $\frac{3}{4}$. Of these, $\frac{3}{4}$ is the larger.

Taking another fraction, say $\frac{5}{9}$ and adding 1 to the numerator and denominator, we get $\frac{6}{10} = \frac{3}{5}$. To check which among $\frac{5}{9}$ and $\frac{3}{5}$ is larger, we need only look at 5×5 and 3×9. So $\frac{5}{9} < \frac{3}{5}$

Is this true for all fractions?

Take $\frac{4}{3}$. Increasing the numerator and denominator by 1 gives $\frac{5}{4}$;

Since $3 \times 5 < 4 \times 4$, we get $\frac{5}{4} < \frac{4}{3}$.

Here things are reversed. What is the difference between $\frac{4}{3}$ and the fractions taken before?

So, what do we guess to happen in general?

We can use algebra to check whether our guess is right.

Suppose that a, b are natural numbers with $a < b$. We want to check whether $\frac{a}{b} < \frac{a+1}{b+1}$. For that, we must look at the products $a\,(b + 1)$ and $b\,(a + 1)$;

$$a\,(b+1) = ab + a$$

$$b\,(a+1) = ba + b$$

In these, we have $ba = ab$. So, what we have to find out is which of $ab + a$, and $ab + b$ is the larger.

b is the larger of a and b. So, what we get by adding the number b to the number ab is larger than what we get by adding a. That is $ab + a < ab + b$; that is

$a\,(b + 1) < b\,(a + 1)$. This gives $\frac{a}{b} < \frac{a+1}{b+1}$.

Now start with $b < a$ and see what you get?

Didn't you get $\frac{a+1}{b+1} < \frac{a}{b}$?

Another problem: we have seen that starting with two equal fractions, we can get another by adding numerators and denominators. What if do this in a pair of unequal fractions?

For example, we know that $\frac{1}{2} < \frac{3}{4}$. Let's add numerator and denominator:

$$\frac{1+3}{2+4} = \frac{4}{6} = \frac{2}{3}$$

We have $\frac{1}{2} < \frac{2}{3}$ and $\frac{2}{3} < \frac{3}{4}$

Thus

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$$

Is this true for all such pairs? Check some more pairs.

Now to see whether this is true in general, we use algebra.

Let's start with $\frac{a}{b} < \frac{p}{q}$. Then $aq < bp$. Now to find the larger of $\frac{a}{b}$ and $\frac{a+p}{b+q}$, we must look at the products $a(b+q)$ and $b(a+p)$.

$$a(b+q) = ab + aq$$

$$b(a+p) = ab + bp$$

Since $aq < bp$, we have $ab + aq < ab + bp$, so that $a(b+q) < b(a+p)$ and so, $\frac{a}{b} < \frac{a+p}{b+q}$. Similarly we get $\frac{a+p}{b+q} < \frac{p}{q}$ (Try it!).

What is the relation between the general principles got from this problem and the one got from the first problem?

Using this idea, we found $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$. Using this again between these fractions, we can see that $\frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{5}{7} < \frac{3}{4}$. And we can continue as much as we like.

(1) Find the larger of each pair of fractions below, without multiplication:

i) $\frac{13}{17}, \frac{14}{15}$ ii) $\frac{13}{17}, \frac{11}{18}$ iii) $\frac{14}{15}, \frac{11}{18}$

(2) Find the larger of each pair of fractions below, without pen and paper:

i) $\frac{3}{5}, \frac{8}{13}$ ii) $\frac{3}{5}, \frac{6}{11}$ iii) $\frac{101}{102}, \frac{98}{99}$

(3) i) Find three fractions larger than $\frac{1}{3}$ and smaller than $\frac{1}{2}$.

ii) Find three such fractions, all with the denominator 24.

iii) Find three such fractions, all with the numerator 4.

(4) From a fraction, a new fraction is formed by adding the same natural number to both the numerator and the denominator.

i) In what kind of fractions does this give a larger fraction?

ii) In what kind of fractions does this give a smaller fraction?

Operations with fractions

What is the sum of $\frac{1}{2}$ and $\frac{1}{3}$?

We first convert them to $\frac{1}{6}$'s, right?

$$\frac{1}{2} = \frac{3}{6}\ ;\ \frac{1}{3} = \frac{2}{6}$$

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

How about $\frac{2}{5} + \frac{3}{7}$?

$$\frac{2}{5} = \frac{14}{35}\ ;\ \frac{3}{7} = \frac{15}{35}$$

$$\frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}$$

We can write in algebra, the general method of adding fractions. Let's take the fractions as $\frac{a}{b}$ and $\frac{p}{q}$. We first put them in forms with the same denominator:

$$\frac{a}{b} = \frac{aq}{bq} \qquad \frac{p}{q} = \frac{bp}{bq}$$

Then we just add the numerators:

$$\frac{a}{b} + \frac{p}{q} = \frac{aq}{bq} + \frac{bp}{bq} = \frac{aq + bp}{bq}$$

For any two fractions $\frac{a}{b}$ and $\frac{p}{q}$,

$$\frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}$$

For example,

$$\frac{3}{4} + \frac{2}{7} = \frac{(3 \times 7) + (4 \times 2)}{4 \times 7} = \frac{29}{28} = 1\frac{1}{28}$$

The difference of two fractions can also be put in this form:

Application and theory

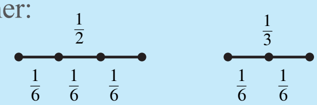
The addition of fractions, written in algebra is

$$\frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}$$

This operation is not very easy. Why is it like this?

General principles about numbers originate from the contexts in which they are used. (Not the other way round). For example, what is the total length of a $\frac{1}{2}$ metre string and $\frac{1}{3}$ metre string joined end to end?

$\frac{1}{2}$ metre is three $\frac{1}{6}$ metres put together. And $\frac{1}{3}$ metre? Two $\frac{1}{6}$ metres put together:



Joining them, we get five $\frac{1}{6}$ metres; that is $\frac{5}{6}$ metres.

So, $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

It is from such contexts that general definitions of additions which hold for all sums arise.

For any fractions $\frac{a}{b}, \frac{p}{q}$ with $\frac{p}{q} < \frac{a}{b}$,

$$\frac{a}{b} - \frac{p}{q} = \frac{aq - bp}{bq}$$

For example,

$$\frac{3}{4} - \frac{2}{7} = \frac{(3 \times 7) - (4 \times 2)}{4 \times 7} = \frac{13}{28}$$

These operations are particularly simple for fractions with numerator 1:

$$\frac{1}{a} + \frac{1}{b} = \frac{(1 \times b) + (1 \times a)}{ab} = \frac{b + a}{ab}$$

and similarly $\frac{1}{a} - \frac{1}{b} = \frac{b - a}{ab}$

For example, $\frac{1}{9} + \frac{1}{11} = \frac{11 + 9}{9 \times 11} = \frac{20}{99}$

$$\frac{1}{9} - \frac{1}{11} = \frac{11 - 9}{9 \times 11} = \frac{2}{99}$$

Let's look at some problems: We can easily see that

$$1 - \frac{1}{2} = \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{3} = \frac{1}{2 \times 3}, \quad \frac{1}{3} - \frac{1}{4} = \frac{1}{3 \times 4}$$

So, we get,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

Another question

An exam paper has two parts, A and B. In part A, there are 2 questions and in part B, 3 questions. A person wrote answers for only one question from each part. Is it correct to say that he did $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ of the exam?

What do we get on adding

$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}$ and so on, up to $\frac{1}{99 \times 100}$?

Now form the first pattern, we also get

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{2 \times 3} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{3 \times 4} = \frac{1}{4} + \frac{1}{12}$$

In other words, we can split any fraction of numerator 1 as the sum of two such fractions. (Fractions of numerator 1 are usually called *unit fractions*).

How do we write a fraction of numerator 2 as the sum of three unit fractions?

Now let's look at multiplication and division of fractions. The product of two fractions is got by multiplying the numerators and denominators separately, right? For example,

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

In general , we can put it like this

For any fractions $\frac{a}{b}$ and $\frac{p}{q}$,

$$\frac{a}{b} \times \frac{p}{q} = \frac{ap}{bq}$$

We have also seen that dividing a fraction by another means, multiplication of the first by the reciprocal of the other.

For example,

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}$$

For any fractions $\frac{a}{b}$ and $\frac{p}{q}$,

$$\frac{a}{b} \div \frac{p}{q} = \frac{aq}{bp}$$

Now some problems.

We have

$$\frac{1}{2} + \frac{1}{2} = 1 \qquad 2 + 2 = 4 \qquad 2 \times 2 = 4$$

Let's take another pair of fractions with sum 1 and check this:

$$\frac{1}{3} + \frac{2}{3} = 1 \qquad 3 + \frac{3}{2} = \frac{9}{2} \qquad 3 \times \frac{3}{2} = \frac{9}{2}$$

One more:

$$\frac{2}{5} + \frac{3}{5} = 1 \qquad \frac{5}{2} + \frac{5}{3} = \frac{25}{6} \qquad \frac{5}{2} \times \frac{5}{3} = \frac{25}{6}$$

Is it true that for any pair of fractions with sum 1, the sum and product of the reciprocals are equal?

Let's use algebra to check. Let's start with $\frac{a}{b} + \frac{p}{q} = 1$.

This gives

$$\frac{aq + bp}{bq} = 1$$

If a fraction equals 1, then its numerator and denominator must be equal. So,

$$aq + bp = bq$$

Now let's compute the sum of the reciprocals:

$$\frac{b}{a} + \frac{q}{p} = \frac{bp + aq}{ap}$$

Since, $aq + bp = bq$, the fraction on the right of this equation becomes $\frac{bq}{ap}$.

Thus we get,

$$\frac{b}{a} + \frac{q}{p} = \frac{bq}{ap}$$

What about the product of the reciprocals?

$$\frac{b}{a} \times \frac{q}{p} = \frac{bq}{ap}$$

So, we have

$$\frac{b}{a} + \frac{q}{p} = \frac{b}{a} \times \frac{q}{p}$$

Find the general principle of each of the patterns below and explain it using algebra.

1) $1 - \frac{1}{3} = \frac{2}{3} = \frac{2}{2^2 - 1}$; $\frac{1}{2} - \frac{1}{4} = \frac{2}{8} = \frac{2}{3^2 - 1}$; $\frac{1}{3} - \frac{1}{5} = \frac{2}{15} = \frac{2}{4^2 - 1}$

2. $\frac{1}{2} + \frac{2}{1} = \frac{5}{2} = 2 + \frac{1}{1 \times 2}$; $\frac{2}{3} + \frac{3}{2} = \frac{13}{6} = 2 + \frac{1}{2 \times 3}$;

$\frac{3}{4} + \frac{4}{3} = \frac{25}{12} = 2 + \frac{1}{3 \times 4}$

3. $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ $\qquad$ $\frac{3}{2} - \frac{2}{3} = \frac{5}{6}$

$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ $\qquad$ $\frac{4}{3} - \frac{3}{4} = \frac{7}{12}$

$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ $\qquad$ $\frac{5}{4} - \frac{4}{5} = \frac{9}{20}$

4. $4\frac{1}{2} - 1\frac{1}{2} = 3$ $\qquad$ $4\frac{1}{2} \div 1\frac{1}{2} = 3$

$5\frac{1}{3} - 1\frac{1}{3} = 4$ $\qquad$ $5\frac{1}{3} \div 1\frac{1}{3} = 4$

$6\frac{1}{4} - 1\frac{1}{4} = 5$ $\qquad$ $6\frac{1}{4} \div 1\frac{1}{4} = 5$

Decimal forms

A fraction of denominator a power of 10 such as 100, 1000 and so on, can be written in shortened form as a decimal.

For example,

$$\frac{3}{10} = 0.3$$

$$\frac{23}{100} = 0.23$$

$$\frac{327}{1000} = 0.327$$

$$\frac{3}{100} = 0.03$$

Some fractions whose denominators are not powers of 10 can also be converted to such a form. For example,

$$\frac{1}{2} = \frac{5}{10} = 0.5$$

$$\frac{1}{5} = \frac{2}{10} = 0.2$$

$$\frac{1}{4} = \frac{25}{100} = 0.25$$

$$\frac{3}{4} = \frac{75}{100} = 0.75$$

Let's look at some more examples. How do we write $\frac{1}{8}$ in the decimal form?

We have $8 = 2^3$; also we can write

$$10^3 = (2 \times 5)^3 = 2^3 \times 5^3$$

That is,

$$1000 = 8 \times 125$$

Form this we get,

$$\frac{1}{8} = \frac{1 \times 125}{8 \times 125} = \frac{125}{1000} = 0.125$$

Similarly,

$$\frac{3}{125} = \frac{3 \times 8}{125 \times 8} = \frac{24}{1000} = 0.024$$

What about $\frac{3}{160}$?

First we factorise the denominator:

$$160 = 32 \times 5 = 2^5 \times 5 = 2^4 \times 2 \times 5 = 2^4 \times 10$$

What power of 10 can we get from this by multiplication?

And what number do we multiply it with?

$2^4 \times 5^4 = 10^4$, right?

So,

$$160 \times 5^4 = (2^4 \times 10) \times 5^4 = 2^4 \times 5^4 \times 10 = 10^5 = 100000$$

And using this, we get

$$\frac{3}{160} = \frac{3 \times 5^4}{160 \times 5^4} = \frac{3 \times 625}{100000} = \frac{1875}{100000} = 0.01875$$

Can we write all fractions in decimal form like this?

How about $\frac{1}{3}$?

Can we get a power of 10 by multiplying 3 by any number?

Putting this in different words, is 3 a factor of any power of 10?

The only prime factors of 10 are 2 and 5 and so these are the only prime factors of any power of 10.

So, 3 is not a factor of any power of 10; and because of this, $\frac{1}{3}$ does not have a form with denominator a power of 10. However, we can do another thing: we can produce a lot of fractions with denominators powers of 10, getting closer and closer to $\frac{1}{3}$. For this, we first write $\frac{1}{3}$ like this:

$$\frac{1}{3} = \frac{1}{10} \times \frac{10}{3}$$

Decimal length

How do we write $\frac{1}{8}$ metres without using fractions, in terms of a smaller unit? Fist we write it in decimetre ($\frac{1}{10}$ metre) $\frac{1}{8}$ metres is $\frac{10}{8}$ decimetres; that is $1\frac{2}{8}$ decimetres. How about changing this $\frac{2}{8}$ decimetres to centimetres? It is $\frac{20}{8}$ centimetres; that is $2\frac{4}{8}$ centimetres. Changing $\frac{4}{8}$ centimetres into millimetres, it becomes $\frac{40}{8} = 5$ millimetres. Thus $\frac{1}{8}$ metre is 1 decimetre, 2 centimetres and 5 millimetres. That is 123 millimetres.

In this, we can write $\frac{10}{3}$ as $3 + \frac{1}{3}$. So we can write the equation above like this:

$$\frac{1}{3} = \frac{1}{10}\left(3 + \frac{1}{3}\right) = \frac{3}{10} + \frac{1}{30}$$

The fraction $\frac{3}{10}$ on the right of this equation has denominator 10 and its difference from $\frac{1}{3}$ is,

$$\frac{1}{3} - \frac{3}{10} = \frac{1}{30}$$

Again we can get a fraction of denominator 100 still closer to $\frac{1}{3}$. For that we should write $\frac{1}{3}$ like this:

$$\frac{1}{3} = \frac{1}{100} \times \frac{100}{3}$$

And if we write $\frac{100}{3}$ on the right side as $33 + \frac{1}{3}$ we get

$$\frac{1}{3} = \frac{1}{100}\left(33 + \frac{1}{3}\right) = \frac{33}{100} + \frac{1}{300}$$

And rewriting this, we find

$$\frac{1}{3} - \frac{33}{100} = \frac{1}{300}$$

Continuing like this, we get

$$\frac{1}{3} - \frac{3}{10} = \frac{1}{30}$$

$$\frac{1}{3} - \frac{33}{100} = \frac{1}{300}$$

$$\frac{1}{3} - \frac{333}{1000} = \frac{1}{3000}$$

$$\frac{1}{3} - \frac{3333}{10000} = \frac{1}{30000}$$

and so on. Thus the difference between the fractions $\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \dots$ from $\frac{1}{3}$ gets smaller and smaller. We can make this difference as small as we wish. Put another way, the fractions continuing as $\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \dots$ get closer and closer to $\frac{1}{3}$.

Unending measurements

How do we write $\frac{1}{3}$ metres without fractions? We can write $\frac{10}{3} = 3\frac{1}{3}$ decimetres. What if we write $\frac{1}{3}$ decimetres in centimetres? $\frac{1}{3}$ decimetre $= \frac{10}{3} = 3\frac{1}{3}$ centimetres. And $\frac{1}{3}$ centimetres in millimetres? It becomes $3\frac{1}{3}$ millimetres. So $\frac{1}{3}$ metre is 3 decimetres, 8 centimetres, 3 millimetres and $\frac{1}{3}$ millimetres more. That is $333\frac{1}{3}$ millimetres. What if we use micrometre, which is $\frac{1}{1000}$ of a millimetre as our unit? $\frac{1}{3}$ metres becomes $333333\frac{1}{3}$ micrometres.

However much we divide into parts of ten, does this end?

We write this fact in short as

$$\frac{1}{3} = 0.333...$$

Note that the decimal form 0.333... given here is different from the decimal forms we have seen so far.

Every decimal form seen earlier is a single fraction with the denominator being a power of ten; but this new form stands for a fraction approached closer and closer by a never ending line of fractions, of denominators powers of ten .

Thus, to include a number like $\frac{1}{3}$ we make a new kind of decimal form.

Let's take another example; $\frac{1}{6}$ which also does not have an equal fraction with denominator a power of 10 (why?). Let's see if we can write it in this new decimal form.

As in the case of $\frac{1}{3}$, we first write $\frac{1}{6}$ like this:

$$\frac{1}{6} = \frac{1}{10} \times \frac{10}{6}$$

Though we can write $\frac{10}{6}$ as $\frac{5}{4}$, we put off simplifications to the end, to make things clear.

Now writing $\frac{10}{6}$ as $1 + \frac{4}{6}$, we get

$$\frac{1}{6} = \frac{1}{10}\left(1 + \frac{4}{6}\right) = \frac{1}{10} + \frac{4}{60}$$

From this, we get

$$\frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$$

Thus we get a fraction of denominator 10, close to $\frac{1}{6}$. Now to get a fraction of denominator 100, closer to $\frac{1}{6}$, start with

$$\frac{1}{6} = \frac{1}{100} \times \frac{100}{6}$$

Writing the $\frac{100}{6}$ on the right side as $16 + \frac{4}{6}$, we get

$$\frac{1}{6} = \frac{1}{100}\left(16 + \frac{4}{6}\right) = \frac{16}{100} + \frac{4}{600}$$

And from this we find,

$$\frac{1}{6} - \frac{16}{100} = \frac{4}{600} = \frac{1}{150}$$

Continuing like this, we get

$$\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$$

$$\frac{1}{6} - \frac{16}{100} = \frac{1}{150}$$

$$\frac{1}{6} - \frac{166}{1000} = \frac{1}{1500}$$

$$\frac{1}{6} - \frac{1666}{10000} = \frac{1}{15000}$$

and so on.

What do we see here?

The fractions $\frac{1}{10}, \frac{16}{100}, \frac{166}{1000}, \ldots$ get closer and closer to $\frac{1}{6}$.

As in the case of $\frac{1}{3}$, we write this fact in short, as a decimal form:

$$\frac{1}{6} = 0.1666\ldots$$

The process of finding this can be simplified as shown below:

	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	
	0.	1	6	6	
6	1.	0	0	0	
		6			
		4	0		$\rightarrow \frac{1}{6} = \frac{1}{10} + \frac{4}{60}$
		3	6		
			4	0	$\rightarrow \frac{1}{6} = \frac{16}{100} + \frac{4}{600}$
			3	6	
				4	$\rightarrow \frac{1}{6} = \frac{166}{1000} + \frac{4}{6000}$

Let's do $\frac{2}{3}$ like this:

	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	
	0.	6	6	6	
3	2.	0	0	0	
	1	8			
		2	0		$\rightarrow \frac{2}{3} = \frac{6}{10} + \frac{2}{30}$
		1	8		
		2	0		$\rightarrow \frac{2}{3} = \frac{66}{100} + \frac{2}{300}$
		1	8		
		2	0		$\rightarrow \frac{2}{3} = \frac{666}{1000} + \frac{2}{3000}$

$$\frac{2}{3} = 0.666\ldots$$

As in the previous examples, this means the fractions $\frac{6}{10}, \frac{66}{100}, \frac{666}{1000}, \ldots$ get closer and closer to $\frac{2}{3}$.

We can note another thing here. Since the remainder is 2 itself from the very first division, we can see that the quotient would be repeatedly 6 thereafter. That is , in this procedure, if a remainder is repeated, then from that place onwards, the digit in the quotient would be repeated.

For example let's take $\frac{1}{7}$.

$$\begin{array}{r|lllllll}
 & 1 & \frac{1}{10} & \frac{1}{10^2} & \frac{1}{10^3} & \frac{1}{10^4} & \frac{1}{10^5} & \frac{1}{10^6} \\
 & 0. & 1 & 4 & 2 & 8 & 5 & 7 \\
\hline
7 & 1. & 0 & 0 & 0 & 0 & 0 & \\
 & & 7 & & & & & \\
\hline
 & & 3 & 0 & & & & \\
 & & 2 & 8 & & & & \\
\hline
 & & & 2 & 0 & & & \\
 & & & 1 & 4 & & & \\
\hline
 & & & & 6 & 0 & & \\
 & & & & 5 & 6 & & \\
\hline
 & & & & & 4 & 0 & \\
 & & & & & 3 & 5 & \\
\hline
 & & & & & & 5 & 0 \\
 & & & & & & 4 & 9 \\
\hline
 & & & & & & & 1
\end{array}$$

$$\frac{1}{7} = \frac{1}{10} + \frac{3}{7 \times 10}$$

$$\frac{1}{7} = \frac{14}{10^2} + \frac{2}{7 \times 10^2}$$

$$\frac{1}{7} = \frac{142}{10^3} + \frac{6}{7 \times 10^3}$$

$$\frac{1}{7} = \frac{1428}{10^4} + \frac{4}{7 \times 10^4}$$

$$\frac{1}{7} = \frac{14285}{10^5} + \frac{5}{7 \times 10^5}$$

$$\frac{1}{7} = \frac{142857}{10^4} + \frac{4}{7 \times 10^4}$$

If we continue, the same digit 1, 4, 2, 8, 5, 7 would be repeated in this order (Why?)

Thus we write

$$\frac{1}{7} = 0.142857142857\ldots$$

Find the decimal forms of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \ldots$ and so on. What is the relation between these and the calculations done in the section, **Cyclic division** of the lesson, **Equal sharing** in the Class 5 textbook?

Now look at this line of numbers:

$$\frac{49}{100}, \frac{499}{1000}, \frac{4999}{10000}, \ldots$$

Do they get closer and closer to any number?

We can easily see that

$$\frac{1}{2} - \frac{49}{100} = \frac{1}{100}$$

$$\frac{1}{2} - \frac{499}{1000} = \frac{1}{1000}$$

$$\frac{1}{2} - \frac{4999}{10000} = \frac{1}{10000}$$

Thus, these numbers get closer and closer to $\frac{1}{2}$. So, according to the new decimal forms, we can also write.

$$\frac{1}{2} = 0.4999...$$

We have already seen the usual decimal form of $\frac{1}{2}$

$$\frac{1}{2} = \frac{5}{10} = 0.5$$

Similarly, we can see that the numbers 0.19, 0.199, 0.1999, ... get closer and closer to $\frac{1}{5}$. Thus $\frac{1}{5}$ has a new decimal form as 0.1999...., in addition to the original form 0.2.

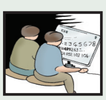
In general, when we create new decimal forms, all the old forms have a new form also.

What about the numbers 0.9, 0.99, 0.999, ... ?

They get closer and closer to 1, right?

So, according to the new representation, we can also write.

$$1 = 0.999...$$



(1) Find the fraction of denominator is a power of 10 equal to each of the fractions below, and then write their decimal forms:

i) $\frac{1}{50}$ ii) $\frac{3}{40}$ iii) $\frac{5}{16}$ iv) $\frac{12}{625}$

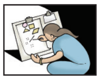
(2) Find fractions of denominators which are powers of 10, getting closer and closer to each of the fractions below and then write their decimal form.

i) $\frac{5}{6}$ ii) $\frac{3}{11}$ iii) $\frac{23}{11}$ iv) $\frac{1}{13}$

(3) i) Explain using algebra, that the fractions $\frac{1}{10}, \frac{11}{100}, \frac{111}{1000}, ...$ gets closer and closer to $\frac{1}{9}$.

ii) Using the general principle above on single digit numbers, find the decimal forms of $\frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}$ (Why $\frac{3}{9}$ and $\frac{6}{9}$ are left out in this?)

iii) What can we say in general about those decimal forms in which a single digit repeats?

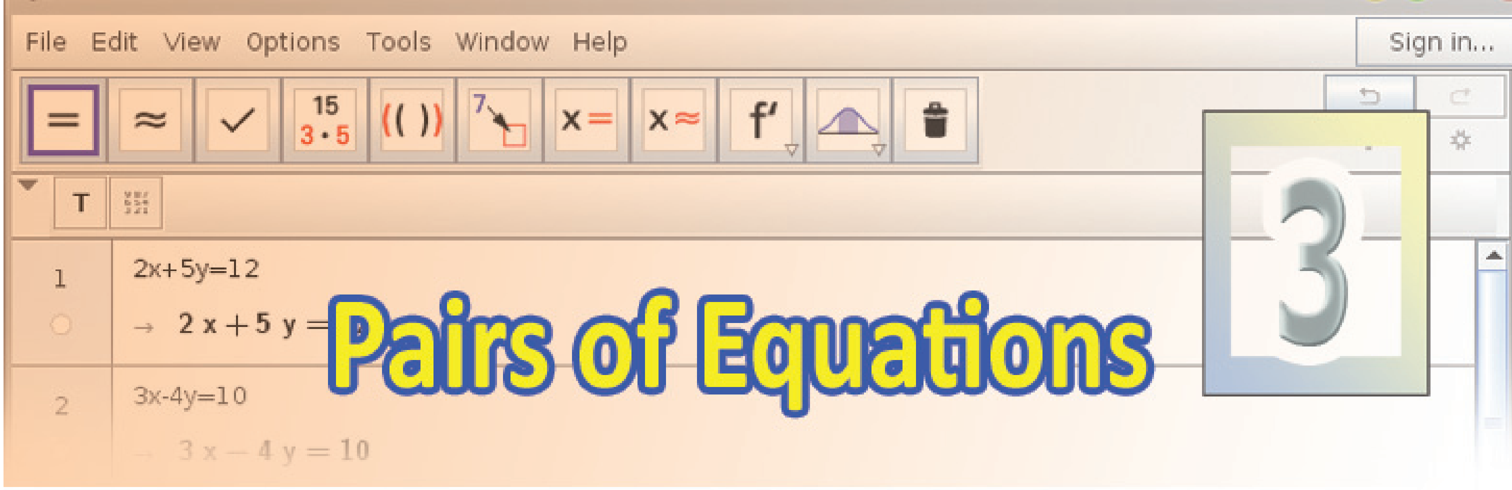


Project

Can every fraction be written as the sum of unit fractions? Justify.

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Explaining different ways of checking the equality of two fractions. • Explaining different ways of finding fractions equal to a specified fraction. • Describing various ways to compare two fractions. • Using algebra to understand and explain patterns of fractions. • Describing the method of writing fractions in the decimal form.			



3 Pairs of Equations

Mental math and algebra

Let's start with a problem:

> There are 100 beads in a box, some black and some white; 10 more black than white. How many black, how many white?

We can do this in several ways. Taking out the 10 extra black beads for the time being, we have 90 left in the box, black and white equal. So, 45 of each. Adding the 10 black beads kept apart, black becomes 55 and white remains at 45.

We can do it with algebra also. (Recall the lesson, **Equations** in Class 8).

Taking the numbers of black beads as x, the number of white beads is $x - 10$ and since there are 100 in all,

$$x + (x - 10) = 100$$

We can extract x from this:

$$\begin{aligned} 2x - 10 &= 100 \\ 2x &= 110 \\ x &= 55 \end{aligned}$$

Thus we find the number of black beads as 55; subtracting 10, we get the number of white beads as 45.

There is another way using algebra again. Taking the number of black beads as x and the number of white beads as y, we can write what we are told as two equations:

$$x + y = 100$$

$$x - y = 10$$

How do we separate x and y from these?

Recall what we learnt about the sum and difference of two numbers, in class 7; adding the sum and difference of two numbers gives twice the larger number. (The section, **Sum and difference** in the lesson, **Unchanging Relations**)

And we also saw that the difference subtracted from the sum gives twice the smaller number.

So in our bead problem,

$$2x = (x + y) + (x - y) = 110$$

$$2y = (x + y) - (x - y) = 90$$

Now we can see that $x = 55$ and $y = 45$.

Here is another problem:

> The price of a table and a chair together is 5000 rupees. The price of a table and four chairs is 8000 rupees. What is the price of each?

Let's first see whether we can do this in head. For a table and four chairs, the price increases by 3000 rupees. It's because of the three extra chairs, isn't it? That is, the extra 3000 rupees is the price of three chairs. So, the price of a chair is 1000 rupees and the price of a table is 4000 rupees.

Instead of thinking it out like this, we can start by taking the price of a chair as x rupees; further thinking gives the price of a table as $5000 - x$ rupees and the price of a table and four chairs as $(5000 - x) + 4x$ rupees. This is said to be 8000 rupees. So,

$$(5000 - x) + 4x = 8000$$

From this, we can find out x

$$5000 + 3x = 8000$$

$$3x = 3000$$

$$x = 1000$$

Thus we find the price of a chair as 1000 rupees; and the price of a table as $5000 - 1000 = 4000$ rupees.

Without thinking any further, we can start by taking the price of a chair as x rupees and the price of a table as y rupees; and then write the given facts as two equations:

$$x + y = 5000$$

$$4x + y = 8000$$

Then we may write y in terms of x using the first equation:

$$y = 5000 - x$$

Now in the second equation, we can write $5000 - x$ in the place of y:

$$4x + (5000 - x) = 8000$$

This is the earlier equation we got, starting with just the price of a chair as x, isn't it? So we can compute the prices as before.

One more problem:

> When we add one to the numerator of a fraction and simplify it, we get $\frac{1}{2}$. When we add one to the denominator instead and simplify it, we get $\frac{1}{3}$. What is this fraction?

This is difficult to do in head. Even by taking just the numerator or denominator as x, we can't go very far. So, let's start by taking the numerator as x and the denominator as y. Then each of the two facts given in the problem can be translated to equations:

$$\frac{x+1}{y} = \frac{1}{2}$$

$$\frac{x}{y+1} = \frac{1}{3}$$

Using cross multiplication (the lesson, **Fractions**) we can put them like this:

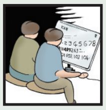
$$2(x + 1) = y$$

$$3x = y + 1$$

The first equation says the number y is equal to the number $2(x + 1)$. So, we can write $2(x + 1)$ for y in the second equation:

$$3x = 2(x + 1) + 1 = 2x + 3$$

From this, we get $x = 3$; then from the first equation, we can find $y = 2 \times 4 = 8$. Thus $\frac{3}{8}$ is the fraction in the problem.



Do each problem below either in your head, or using an equation with one letter, or two equations with two letters:

(1) In a rectangle of perimeter one metre, one side is five centimetres longer than the other. What are the lengths of the sides?

(2) A class has 4 more girls than boys. On a day when only 8 boys were absent, the number of girls was twice that of boys. How many girls and boys are there in the class?

(3) A man invested 10000 rupees, split into two schemes, at annual rates of interests 8% and 9%. After one year he got 875 rupees as interest from both. How much did he invest in each?

(4) A three and a half metres long rod is to be cut into two pieces, one piece is to be bent into a square and the other into an equilateral triangle. The length of a side of both must be the same. How should it be cut?

(5) The distance travelled in t seconds by an object starting with a speed of u metres/second and moving along a straight line with speed increasing at the rate of a metres/second every second is given by $ut + \frac{1}{2}at^2$ metres. An object moving in this manner travels 10 metres in 2 seconds and 28 metres in 4 seconds. With what speed did it start? At what rate does its speed change?

Two equations

See this problem:

> The price of 2 pens and 3 notebooks is 40 rupees; and the price of 2 pens and 6 notebooks is 60 rupees. What is the price of a pen? And the price of a notebook?

Recall how we first solved the table and chair problem. Why does the price increase from 40 to 60 here?

Because of 2 more notebooks, right? In other words, the increase of 20 rupees is the price of 2 notebooks. So the price of a notebook is 10 rupees.

Now to get the price of 2 pens, we need only subtract the price of the 3 notebooks from 40 rupees, right? That is, 40 – 30 = 10 rupees. So, the price of a pen is 5 rupees.

Let's see how we can do this by taking the price of a pen as x rupees, the price of a notebook as y rupees and then writing the given facts as equations:

The price of 2 pens and 3 notebooks is 40 rupees	$2x + 3y = 40$
The price of 2 pens and 5 notebooks is 60 rupees	$2x + 5y = 60$
The increase is due to 2 extra notebooks	$(2x + 5y) - (2x + 3y) = 2y$
The increase is 20 rupees	$60 - 40 = 20$
The price of 2 notebooks is 20 rupees	$2y = 20$
The price of a notebook is 10 rupees	$y = 10$
The price of 2 pens is 30 rupees subtracted from 40 rupees	$2x = 40 - (3 \times 10) = 10$
The price of a pen is 5 rupees	$x = 5$

Look at a slightly different problem:

The price of 3 pencils and 4 pens is 26 rupees; and for 6 pencils and 3 pens, it is 27 rupees. What is the price of each?

Let's first try to do this without algebra. In this, the increase in price is not due to the increase in one thing, as in the first problem. So, it is not as easy as that.

If the number of pencils or pens were the same in both the given facts, we could have done it as before. How about making it so?

Let's write the prices like this:

Pencil	Pen	Price
3	4	26
6	3	27

The number of pencils is 3 in the first row and 6 in the second. Can we make it 6 in the first row also?

How about 6 pencils and 8 pens?

	Pencil	Pen	Price
×2 (from row 1 to row 3)	3	4	26
	6	3	27
	6	8	52

The increase of 25 rupees from the second to the third is due to just 5 pens, isn't it?

So, the price of a pen is 5 rupees. Now from the first row, we can compute the price of 3 pencils as $26 - 20 = 6$ rupees and hence the price of a pencil is 2 rupees.

Now let's write all these thoughts in algebra. Taking the price of a pencil as x rupees and the price of a pen as y rupees, we can write the given facts and the method of calculating the prices, like this:

The price of 3 pencils and 4 pens is 26 rupees	$3x + 4y = 26$
The price of 6 pencils and 3 pens is 27 rupees	$6x + 3y = 27$
The price of 6 pencils and 8 pens is 52 rupees	$6x + 8y = 2(3x + 4y) = 52$
The increase is the price of 5 pens	$(6x + 8y) - (6x + 3y) = 5y$
The increase is 25 rupees	$5y = 25$
The price of a pen is 5 rupees	$y = 5$
The price of 3 pencils is 20 rupees subtracted from 26 rupees	$3x = 26 - (4 \times 5) = 6$
The price of a pencil is 2 rupees	$x = 2$

We can rewrite these as follows. First we write the facts given in the problem as equations and label them as equation (1) and equation (2).

$$3x + 4y = 26 \qquad (1)$$

$$6x + 3y = 27 \qquad (2)$$

Equation (1) says, the number $3x + 4y$ is 26; so twice this number is 52.

$$6x + 8y = 52 \qquad (3)$$

Now using equation (2) and Equation (3), we get

$$(6x + 8y) - (6x + 3y) = 52 - 27$$

Simplifying this, we get

$$5y = 25$$

and this gives $y = 5$. Now taking y as 5 in Equation (1), we can compute x:

$$3x + (4 \times 5) = 26$$

$$3x = 26 - 20 = 6$$

$$x = 2$$

Another problem:

> Five small buckets and two large buckets of water make 20 litres; two small buckets and 3 large buckets make only 19 litres. How much water can each bucket hold?

Taking a small bucketful as x litres and a large bucketful as y litres, we can write the given facts as equations:

$$5x + 2y = 20 \qquad (1)$$

$$2x + 3y = 19 \qquad (2)$$

Proceeding as in the first problem, to get $2x$ in equation (1) also, we must multiply by $\frac{2}{5}$ or to get $5x$ in equation (2) also, we must multiply by $\frac{5}{2}$.

Different Facts

Ramu bought a pencil and a pen for 7 rupees. Aju bought 4 pencils and 4 pens for 28 rupees. They tried to calculate the price of each using these facts. Taking the price of a pencil as x rupees, they used the first fact to get the price of a pen as $7 - x$ rupees. Using this in the second fact, they got

$$4x + 4(7 - x) = 28$$

What did they get on simplification? $28 = 28$

What if they had taken the price of a pencil as x rupees and the price of a pen as y rupees?

$$x + y = 7$$

$$4x + 4y = 28$$

If the second equation is written as

$$4(x + y) = 28$$

They would only get

$$x + y = 7$$

again.

In this problem, only one fact is actually given, though stated in two different ways. And using that alone, we cannot find the separate prices.

> **Math and fact**
>
> A rectangle of perimeter 10 metres is to be made with one side 5.5 metres longer than the other.
>
> Taking the length of the shorter side as x metres, the length of the longer side must be $x + 5.5$ metres. Since the perimeter is to be 10 metres.
>
> $$x + (x + 5.5) = \frac{10}{2} = 5$$
>
> That is,
>
> $$2x + 5.5 = 5$$
>
> Which gives
>
> $$2x = -0.5$$
> $$x = -0.25$$
>
> But the length of a side of a rectangle cannot be a negative number.
>
> What this means is that we cannot draw a rectangle satisfying these conditions. In this problem, if we take the length of the sides as x and y metres, we get
>
> $$x + y = 5$$
> $$y - x = 5.5$$
>
> And we can immediately see that there are no positive numbers satisfying both these conditions. (The sum of two positive numbers cannot be less than their difference, right?)

Thus, we can find the prices. There is a way to do this without fractions. We can make $10x$ in both equations; we need only multiply equation (1) by 2 and equation (2) by 5. The equations change like this:

$$(1) \times 2: \quad 10x + 4y = 40 \qquad (3)$$

$$(2) \times 5: \quad 10x + 15y = 95 \qquad (4)$$

Now subtracting equation (3) from equation (4), we get

$$(4) - (3): \quad 11y = 55$$

and this gives

$$y = 5$$

Now using this in Equation (1), we can calculate x:

$$5x + 10 = 20$$

$$5x = 10$$

$$x = 2$$

Thus we see that the small bucket holds 2 litres and the large bucket, 5 litres.

(1) Raju bought seven notebooks of two hundred pages and five of hundred pages, for 107 rupees. Joseph bought five notebooks of two hundred pages and seven of hundred pages, for 97 rupees. What is the price of each kind of notebook?

(2) Four times a number and three times a number added together make 43. Two times the second number, subtracted from three times the first give 11. What are the numbers?

(3) The sum of the digits of a two-digit number is 11. The number got by interchanging the digits is 27 more than the original number. What is this number?

(4) Four years ago, Rahim's age was three times Ramu's age. After two years, it would just be double. What are their ages now?

(5) If the length of a rectangle is increased by 5 metres and breadth decreased by 3 metres, the area would decrease by 5 square metres. If the length is increased by 3 metres and breadth increased by 2 metres, the area would increase by 50 square metres. What are the length and breadth?

Some Other equations

See this problem:

Of two squares, sides of the larger are 5 centimetres longer than smaller and the area of the larger is 55 square centimetres more. What is the length of the sides of each?

Taking the length of a side of the larger square as x centimetres and that of the smaller as y centimetres, we can put the facts given as two equations:

$$x - y = 5$$
$$x^2 - y^2 = 55$$

What do we do next?

Recall that $x^2 - y^2 = (x + y)(x - y)$; this we can write as

$$x + y = \frac{x^2 - y^2}{x - y}$$

So in our problem,

$$x + y = \frac{55}{5} = 11$$

Now we have the sum $x + y = 11$ and the difference, $x - y = 5$. We can calculate the numbers as

$$x = \frac{1}{2}(11 + 5) = 8$$
$$y = \frac{1}{2}(11 - 5) = 3$$

Thus the lengths of the sides of the squares are 8 centimetres and 3 centimetres.

Another problem:

The perimeter of a rectangle is 10 metres and its area is $5\frac{1}{4}$ square metres. What are the lengths of its sides?

Taking the lengths of the sides as x centimetres and y centimetres, perimeter is $2(x+y)$ centimetres and area is xy square centimetres. So the facts given can be written as the equations,

$$x + y = 5$$

$$xy = 5\frac{1}{4}$$

What next? Can we find $x - y$ from these?

Recall that $(x+y)^2 - (x-y)^2 = 4xy$. We can write this as

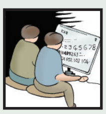
$$(x-y)^2 = (x+y)^2 - 4xy$$

So in our problem

$$(x-y)^2 = 5^2 - \left(4 \times 5\frac{1}{4}\right) = 25 - 21 = 4$$

This gives $x - y = 2$. Together with $x + y = 5$, we can find $x = 3\frac{1}{2}, y = 1\frac{1}{2}$

Thus the lengths of the sides of the rectangle are $3\frac{1}{2}$ metres and $1\frac{1}{2}$ metres.



(1) A 10 metre long rope is to be cut into two pieces and a square is to be made using each. The difference in the areas enclosed must be $1\frac{1}{4}$ square metres. How should it be cut?

(2) The length of a rectangle is 1 metre more than its breadth. Its area is $3\frac{3}{4}$ square metres. What are its length and breadth?

(3) The hypotenuse of a right triangle is $6\frac{1}{2}$ centimetres and its area is $7\frac{1}{2}$ square centimetres. Calculate the lengths of its perpendicular sides.

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• From two given facts about two measures, calculating the measures either mentally or writing the facts as a single equation with a single letter or writing them as two equations with two letters.			

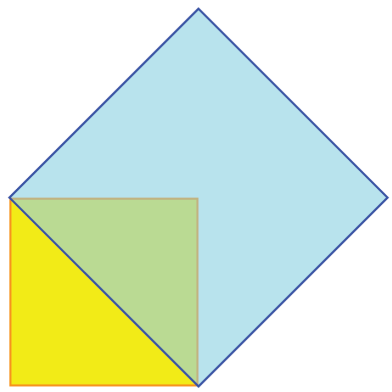
4 New Numbers

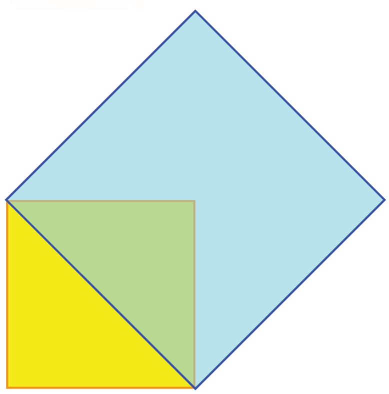
Length and numbers

See the picture:

A small square and a larger one drawn on its diagonal.

We have seen in Class 7 that the area of the large square made thus is twice the area of the smaller one, remember?





That is, if the sides of the small square are one metre long then the area of the large square is two square centimetres.

What is the length of its sides?

It is longer than one metre anyway; and shorter than two metres (how come?) It may be some fraction of a metre between one and two, but then the area of the square being two metres, the square of this fraction must be two.

What fraction has the square two?

One and a half perhaps?

$$\left(1\frac{1}{2}\right)^2 = \left(1+\frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2\frac{1}{4}$$

So, one and a half is a bit too large. One and a quarter?

$$\left(1\frac{1}{4}\right)^2 = 1 + \frac{1}{2} + \frac{1}{16} = 1\frac{9}{16}$$

But now it is smaller than two. One and a third?

$$\left(1\frac{1}{3}\right)^2 = 1 + \frac{2}{3} + \frac{1}{9} = 1\frac{7}{9}$$

It also is smaller, but better that one and a quarter.

Instead of proceeding by trial and error like this, why not try algebra?

Taking x as the numerator and y as the denominator of a fraction whose square is to be 2, we get

$$\left(\frac{x}{y}\right)^2 = 2$$

That is,

$$\frac{x^2}{y^2} = 2$$

We can write it as $x^2 = 2y^2$

Now it is a problem of natural numbers: find two natural numbers, the square of one being double the square of the other. In this, x^2 is a multiple of 2; that is an even number. What about x?

Squares of odd numbers are odd and squares of even numbers are even, right? So, x is also an even number.

It means we can write $x = 2u$. This gives $y^2 = 2u^2$.

Now y^2 and hence y is an even number. So we can write $y = 2v$.

The equation $y^2 = 2u^2$ becomes $(2v)^2 = 2u^2$, and simplifying this we get $2v^2 = u^2$. That is,

$$u^2 = 2v^2$$

It's our old equation again, isn't it? Instead of the numbers x and y we now have u and v. So, everything we have found out about x and y can be repeated for u and v also. And finally concluding that u and v are even numbers and taking

$$u = 2s \qquad v = 2t$$

We would get $s^2 = 2t^2$

This can be continued on and on, but that doesn't solve our problem. We can note another thing here. We got $x = 2x$ and in it, $u = 2s$, so that $x = 4s$. Similarly $y = 4t$. So a and y are not only multiples of 2 as seen earlier, but in fact multiples of 4.

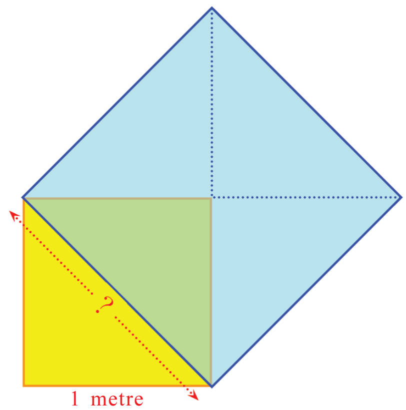
If we continue playing with equations, we would find x and y to be multiples of 8 and 16 and so on.

Thus if there are natural numbers x and y satisfying the equation $x^2 = 2y^2$, then they must be multiple of all powers of 2.

How can that be? Is there any natural number which is a multiple of all power of 2? Thus, there are no natural numbers satisfying our equation and so, no fraction whose square is 2.

The square of any fraction is not 2.

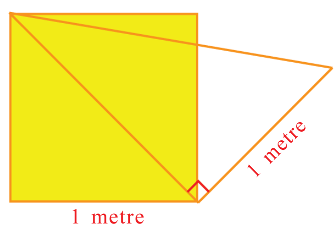
So what happens to our geometric problem?



If the diagonal of a square of side one metre is a fractional multiple of one metre, then the square of the fraction must be two (we have seen in Class 6 that the area of a square is the square of its side, even when the length of a side is a fractional multiple of the unit). But there is no fraction whose square is 2. So, what can we say?

The diagonal of a square of side 1 cannot be expressed as a fraction.

We have several such lengths which cannot be expressed as a natural number or fraction. For example, see this picture.



Origin of numbers

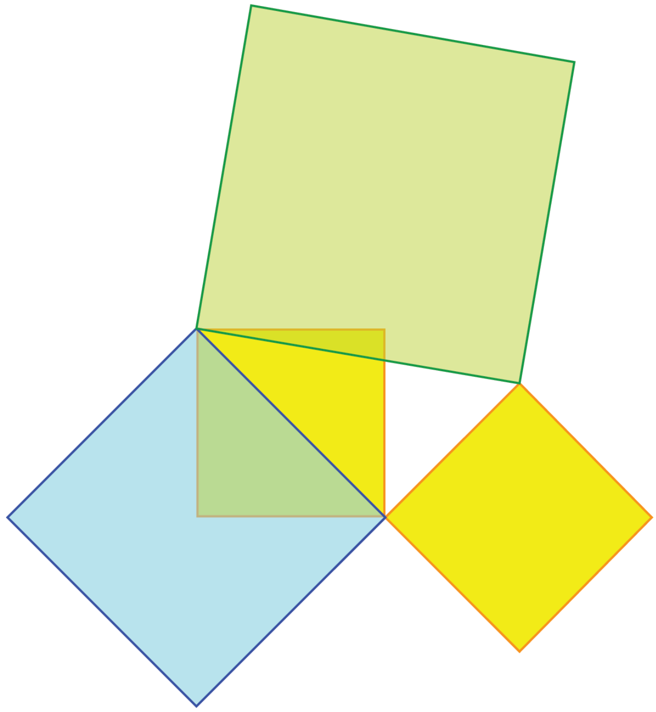
Measure everything and make it a number; and through such numbers and relations between them try to understand the world – this is the basic function of mathematics

Depending on the nature of the things measured, different kinds of numbers need to be created. During the age when men gathered food directly from nature, they needed numbers only for counting – the number of men in a group, the number of cattle in a herd and so on. In other words, only natural numbers were needed at that time.

Later around five thousand BC, as men settled along river banks and started large scale agriculture, they needed to measure all kinds of lengths and areas to mark land and build houses. The concept of fractions arose in this period. New numbers were needed when it was realised that not all measurements could be expressed as fractions.

Still later other kinds of numbers like negative numbers and complex numbers were created for mathematical convenience, rather than physical necessities. That such numbers were also found useful in science like physics is another matter.

What is the hypotenuse of the right triangle drawn on the diagonal of the square? Let's draw squares on all its sides:



By Pythagoras theorem the area of the green square on the hypotenuse of our right triangle is $1 + 2 = 3$. So, if its length is a fractional multiple of 1 metre, then the square of this fraction must be 3.

Just as we proved that the square of any fraction cannot be 2, we can prove that the square of any fraction cannot be 3 either. So, the length of the hypotenuse of this right triangle also cannot be expressed as a fraction.

Let's look at another example: Suppose we want to make a cube of volume 2 cubic centimetres, what should be the length of its sides? We can show that the cube (third power) of any fraction is not 2. So, the length of a side of this cube cannot also be expressed as a fraction.

Thus, there are many instances where we need lengths which cannot be expressed as fractions.

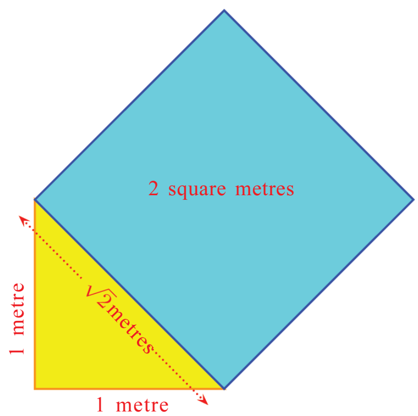
Measures and numbers

We have to make a new kind of number to denote measures which cannot be expressed as fractions. Let's take our first example. How do we denote the length of the diagonal of a square of side 1 (metre or centimetre or whatever)?

We can put the question like this: how do we denote the side of a square of area 2?

For a square with length of a side denoted by a natural number or a fraction, this length is the square root of the area. For example, the length of a side of a square of area 4 is $\sqrt{4} = 2$; if the area is $2\frac{1}{4}$, then the length of a side is $\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$.

Like this, we write the length of a side of a square of area 2 as $\sqrt{2}$:



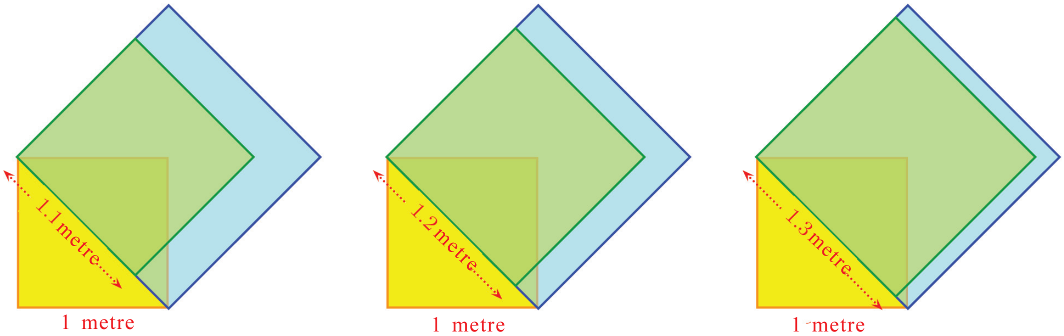
Just giving a symbol to denote the length doesn't solve our problem. To know its size, we must compare it with known lengths. We do this by finding fractions closer and closer to this length. If we mark such lengths along the diagonal, the squares with sides of these lengths get closer and closer to the square on the diagonal.

Crumbling beliefs

During the sixth century BC, Pythagoras and his disciples believed that all measures could be compared using natural numbers. More precisely, they thought that any two measures could be compared by ratios of natural numbers. But the ratio of the diagonal of a square to its side cannot be expressed as a ratio of natural numbers. For if this ratio could be expressed as $a : b$, where a and b are natural numbers, then the diagonal would be $\frac{a}{b}$ times the side, so that the square on the diagonal would be $\frac{a^2}{b^2}$ times the square on the side, which would give $\frac{a^2}{b^2} = 2$. And we have seen that this is impossible.

It is believed that this was discovered by Hippasus, who was a disciple of Pythagoras.

Measures like the diagonal and side of a square, which cannot be compared using ratios of natural numbers, are called *incommensurable magnitudes*.



In terms of numbers alone, this means the (numerical) squares of the fractions expressing the lengths of the sides of these (geometric) squares get closer and closer to 2.

To find such numbers, it is more convenient to use the decimal forms of fractions. Computing first the squares of the fractions 1.1, 1.2, 1.3, ... we find

$$1.4^2 = 1.96$$
$$1.5^2 = 2.25$$

So, taking up to tenths, we get

$$1.4^2 < 2 < 1.5^2$$

Now computing the squares of the numbers 1.41, 1.42, 1.43, between 1.4 and 1.5, we find

$$1.41^2 = 1.9881; \quad 1.42^2 = 2.0164$$

So, taking up to hundredths, we get

$$1.41^2 < 2 < 1.42^2$$

Continuing like this, we find

1.4^2	=	1.96	1.5^2	=	2.25
1.41^2	=	1.9881	1.42^2	=	2.0164
1.414^2	=	1.999396	1.415^2	=	2.002225
1.4142^2	=	1.99996164	1.4143^2	=	2.00024449
1.41421^2	=	1.9999899241	1.41422^2	=	2.0000182084

and so on. That is, up to five decimal places,

$$1.41421^2 < 2 < 1.41422^2$$

We can also note that

$$2 - 1.41421^2 = 0.0000100759 < 0.00002$$

Summarising, we have found this:

The squares of the fractions continuing as $\frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots$ get closer and closer to 2.

We write this fact in short like this:

$$\sqrt{2} = 1.41421\ldots$$

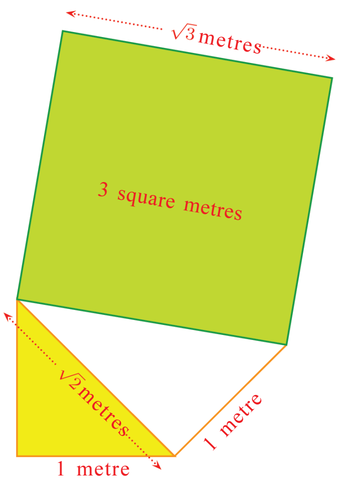
So, we can say that the number $\sqrt{2}$ is 1.4 up to one decimal place, 1.41 upto two decimal places, and so on. We write

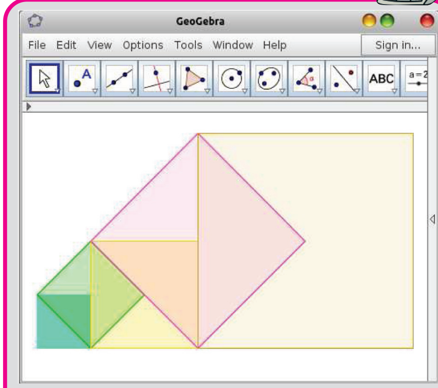
$$\sqrt{2} \approx 1.4$$

$$\sqrt{2} \approx 1.41$$

and so on. The symbol ≈ in this means approximately (nearly) equal.

Like this, the length of the sides of a square of area 3 is said to be $\sqrt{3}$.





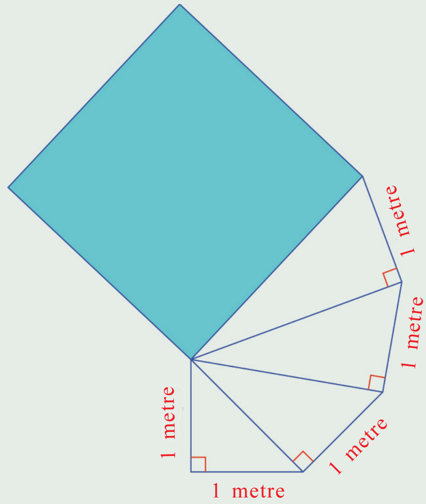
In the picture, a side of the smallest square is 1 centimetre. Calculate the area and a side of the largest square. Draw this picture in GeoGebra, using **Regular Polygon**. Use **Area** to find the area of each square. Which of the squares have a fraction as the length of a side?

Making computations as before, we can see that the squares of the fractions 1.7, 1.73,1.732... get closer and closer to 3. We shorten this by writing $\sqrt{3} = 1.73205\ldots$

In general, for any positive number, the length of a side of a square of area x is written $\sqrt{x}$. In some cases, $\sqrt{x}$ would be a natural number or fraction; in other cases, we compute fractions whose squares get closer and closer to x and write $\sqrt{x}$ in decimal form.

(1) In the picture, the square on the hypotenuse of the top most right triangle is drawn.

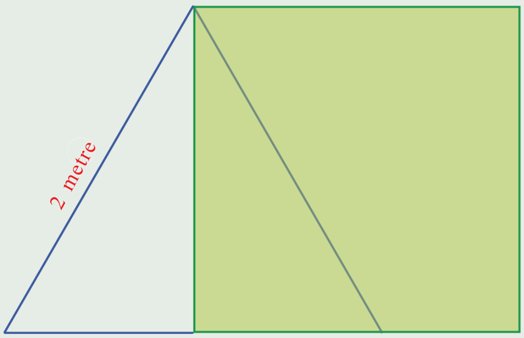
Calculate the area and the length of a side of the square.



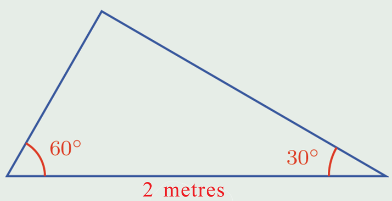
(2) A square is drawn on the altitude of an equilateral triangle of side 2 metres.

i) What is the area of the square?

ii) What is the altitude of the triangle?



iii) What are the lengths of the other two sides of the triangle shown below?



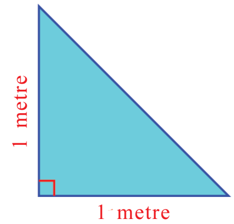
(3) We have seen in Class 7 that any odd number can be written as the difference of two perfect squares. (The lesson, **Identities**) Using this, draw lines of lengths $\sqrt{7}$ and $\sqrt{11}$ centimetres.

(4) Explain two different methods of drawing a line of length $\sqrt{13}$ centimetres.

(5) Find three fractions larger than $\sqrt{2}$ and less than $\sqrt{3}$.

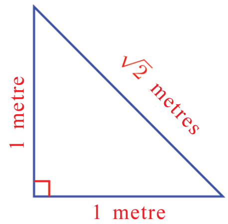
Addition and subtraction

What is the area of a right triangle with lengths of perpendicular sides 1 metre?



And the perimeter?

The length of its hypotenuse is $\sqrt{2}$ metres.

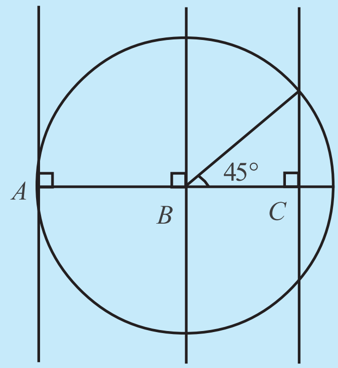


So, to get the perimeter, we must add 2 metres and $\sqrt{2}$ metres. We write it as $2+\sqrt{2}$ metres.

The fractions which are approximately equal to $\sqrt{2}$ continue as 1.4, 1.41, 1.414, ... So the fractions approximately $2+\sqrt{2}$ are got by adding 2 to these; that 3.4, 3.41, 3.414, ...

A different ratio

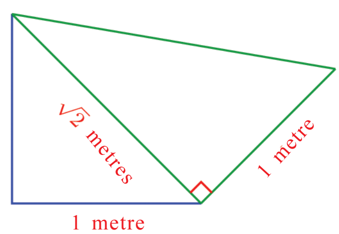
B is the centre of the circle in the picture.



$AB : BC = \sqrt{2} : 1$

In this problem, if we need only measurements correct to a centimetre, we can take the perimeter as 3.41 metres. And if we want accuracy up to millimetres, we take it as 3.414 metres.

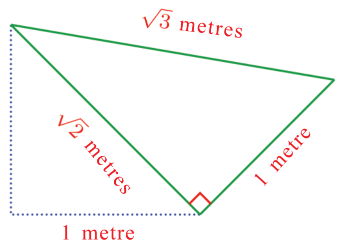
Suppose we draw another right triangle as in the picture, with the hypotenuse of our first triangle as base.



We have seen that its third side is $\sqrt{3}$ metres.

We can write its perimeter as $1+\sqrt{2}+\sqrt{3}$ metres.

To get fractions approximately equal to $\sqrt{2}+\sqrt{3}$, we add approximately equal fractions of each.



$\sqrt{2}$	:	1.4	1.41	1.414
$\sqrt{3}$	:	1.7	1.73	1.732
$\sqrt{2}+\sqrt{3}$	:	3.1	3.14	3.146

Adding 1 to these, we get fractions approximately equal to $1+\sqrt{2}+\sqrt{3}$.

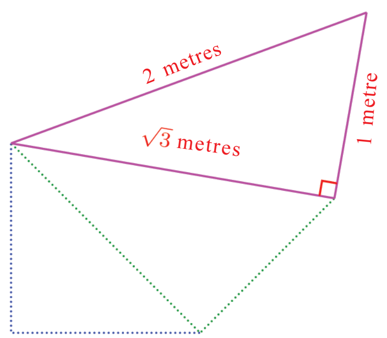
Thus the perimeter of the new triangle, correct to a millimetre is 4.146 metres.

How much more is the perimeter of this triangle, than the perimeter of the first one? We can say, approximately $4.146 - 3.414 = 0.732$ metre.

Or we can compute it like this:

$(1+\sqrt{2}+\sqrt{3}) - (2+\sqrt{2}) = 1+\sqrt{3}-2 = \sqrt{3}-1 \approx 0.732$

Now we draw one more triangle on top of this.



How much more is its perimeter, than that of the second triangle?

The perimeter of this new triangle is $2+1+\sqrt{3} = 3+\sqrt{3}$ metres. We can calculate the increase in perimeter without calculating fractions approximately equal to its perimeter.

The perimeter of the second triangle is $1+\sqrt{2}+\sqrt{3}$ metres. So, the increase in perimeter is

$$(3+\sqrt{3}) - (1+\sqrt{2}+\sqrt{3}) = 2-\sqrt{2}$$

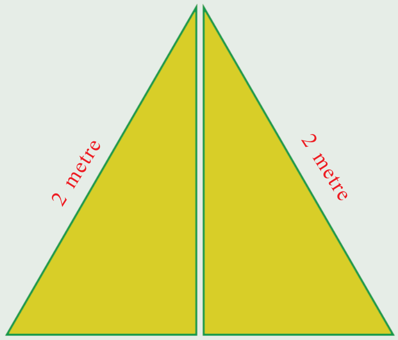
We can compute this up to three decimal places as

$$2 - 1.414 = 0.586$$

So, the perimeter is approximately 0.586 metre more.

(1) The hypotenuse of a right triangle is $1\frac{1}{2}$ metres and another side is $\frac{1}{2}$ metre. Calculate its perimeter correct to a centimetre.

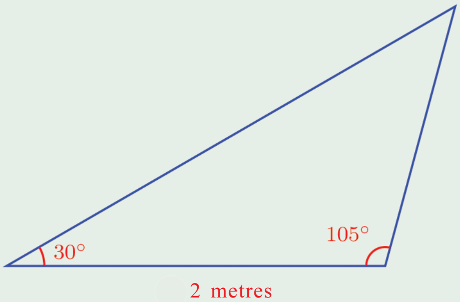
(2) The picture shows an equilateral triangle cut into halves by a line through a vertex.



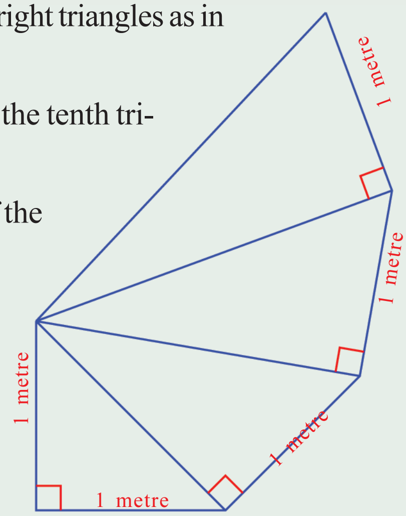
i) What is the perimeter of a part?

ii) How much less than the perimeter of the whole triangle is this?

(3) Calculate the perimeter of the triangle shown below.



(4) We have seen how we can draw a series of right triangles as in the picture.



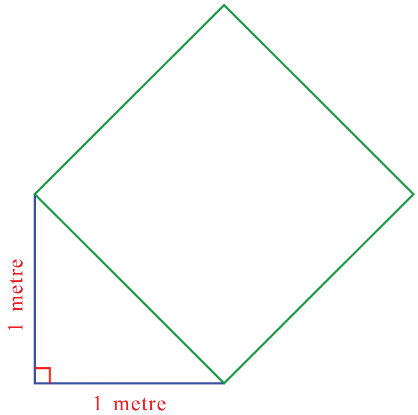
i) What are the lengths of the sides of the tenth triangle?

ii) How much more is the perimeter of the tenth triangle than the perimeter of the ninth triangle?

(iii) How do we write in algebra, the difference in perimeter of the n^{th} triangle and that of the triangle just before it?

(5) What is the hypotenuse of the right triangle with perpendicular sides $\sqrt{2}$ centimetres and $\sqrt{3}$ centimetres? How much larger than the hypotenuse is the sum of the perpendicular sides?

Multiplication



We have seen this picture many times. What is the perimeter of the square in this?

We know that the length of each of its sides is $\sqrt{2}$ metres. So to get the perimeter, we must calculate four times this.

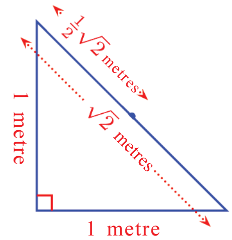
As with other numbers, we write 4 times $\sqrt{2}$ as $4 \times \sqrt{2}$. Usually we write this without the multiplication sign, as $4\sqrt{2}$.

To get fractions approximately equal to this, we find four times the fractions approximately equal to $\sqrt{2}$.

So the perimeter of our square, correct to a millimetre, is

$$4 \times 1.414 = 5.656 \text{ metres}$$

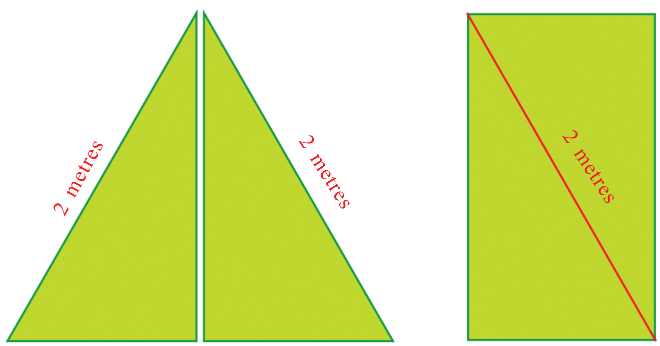
Similarly, we write half of $\sqrt{2}$ as $\frac{1}{2}\sqrt{2}$.



By taking half of fractions approximately equal to $\sqrt{2}$, we get fractions approximately equal to $\frac{1}{2}\sqrt{2}$.

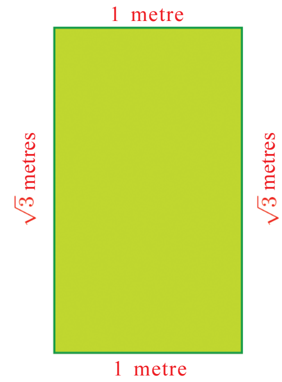
That is $\frac{1}{2}\sqrt{2} = 0.7071\ldots$

Now see these pictures:



An equilateral triangle is cut into halves and the pieces rearranged to make a rectangle.

What is the perimeter of this rectangle?



Since the right triangles are equal, the base of each is 1 metre and we have seen in an earlier problem that the height is $\sqrt{3}$ metres.

So, the perimeter is $2\sqrt{3} + 2$ metres.

We can calculate fractions approximately equal to this numbers as below:

$\sqrt{3}$:	1.7	1.73	1.732 . . .
$2\sqrt{3}$:	3.4	3.46	3.464 . . .
$2\sqrt{3}+2$:	5.4	5.46	5.464 . . .

As with other numbers, is $2\sqrt{3} + 2$ equal to $2(\sqrt{3} + 1)$?

We can compute fractions approximately equal to the second number like this:

$\sqrt{3}$:	1.7	1.73	1.732 . . .
$\sqrt{3} + 1$:	2.7	2.73	2.732 . . .
$2\left(\sqrt{3}+1\right)$:	5.4	5.46	5.464 . . .

Thus the fractions approximating $2\sqrt{3} + 2$ and $2(\sqrt{3} + 1)$ are the same. So,

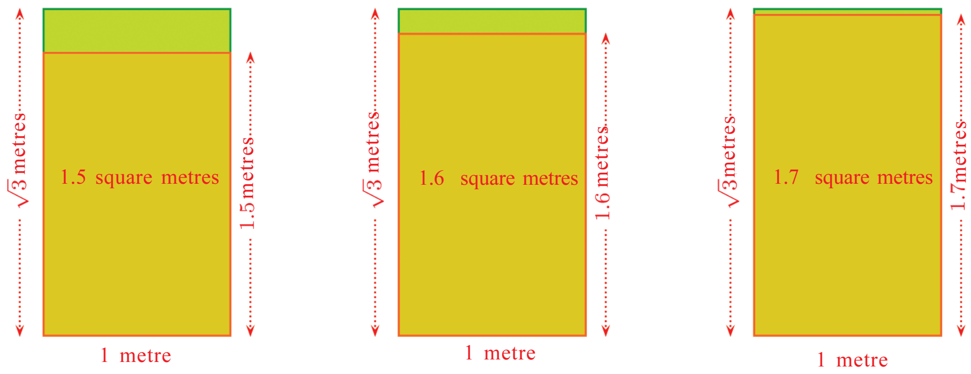
$$2\sqrt{3} + 2 = 2(\sqrt{3} + 1)$$

Now let's see what the area of this rectangle is.

If the lengths of sides are fractions, then the area is their product.

Here also, is the area $1 \times \sqrt{3} = \sqrt{3}$, which is the product of the sides?

To see this, let's draw rectangles of one side 1 metre and the other side fractional lengths approximately equal to $\sqrt{3}$ metres, within our rectangle (as we have done sometime before).



As we continue taking the heights of the inner rectangles as 1.73, 1.432 ... metres, their areas are also the same number in square metres.

Thus the area of a rectangle of sides 1 metre and $\sqrt{3}$ metres is indeed $\sqrt{3}$ square metres.

Now, what if the lengths of the sides are $\sqrt{3}$ and $\sqrt{2}$? We denote the area as $\sqrt{3}\times\sqrt{2}$.

To explain it in terms of numbers, we multiply fractions approximately equal to $\sqrt{3}$ and $\sqrt{2}$ in order and take as many decimal places as we need:

$\sqrt{3}$	:	1.7	1.73	1.732	1.7320	1.73205 . . .
$\sqrt{2}$	:	1.4	1.41	1.414	1.4142	1.41421 . . .
$\sqrt{3}\times\sqrt{2}$	:	2.4	2.44	2.449	2.4494	2.44948 . . .

$$\sqrt{3}\times\sqrt{2} = 2.44948\ \ldots$$

Decimal Math

In shortening numbers in decimal form up to a definite place, if the digit in the next place is more than 5, we increase the digit in the desired place by 1. For example, $1.7 \times 1.4 = 2.38$; and its approximation upto the first decimal place is 2.4.

We note another thing here. The numbers 1.4^2, 1.41^2, 1.414^2, 1.4142^2, . . . get closer and closer to 2. (The very meaning of writing $\sqrt{2} = 1.41421\ldots$ is this).

And the numbers 1.7^2, 1.73^2, 1.732^2, 1.7320^2, 1.73205^2, . . . get closer and closer to 3.

So, the product of these squares must get closer and closer to 6, right?

And the product of squares of fractions is the square of the product.

Thus we have

$$1.7^2 \times 1.4^2 = (1.7 \times 1.4)^2$$

$$1.73^2 \times 1.41^2 = (1.73 \times 1.41)^2$$

$$1.732^2 \times 1.414^2 = (1.732 \times 1.414)^2$$

and so on. In this, the products 1.7×1.4, 1.73×1.41 and so on are already calculated in the last row of our table. So, we can write fractions approximating 2, 3 and 6 like this:

3	:	1.7^2	1.73^2	1.732^2	1.7320^2	1.73205^2 . . .
2	:	1.4^2	1.41^2	1.414^2	1.4142^2	1.41421^2 . . .
6	:	2.4^2	2.44^2	2.449^2	2.4494^2	2.44948^2 . . .

What does the last row of this mean?

The squares of the numbers 2.4, 2.44, 2.449, 2.4494, 2.44948, . . . get closer and closer to 6.

By our definition of new numbers, we write this as

$$\sqrt{6} = 2.44948\ldots$$

We have seen that $\sqrt{3} \times \sqrt{2}$ is also equal to this. So we get,

$$\sqrt{3} \times \sqrt{2} = \sqrt{6}$$

We can see that for numbers other than 2 and 3 also, the product of square roots is equal to the square root of the product (we have already seen in Class 7 that this is so in the case where the square roots are natural numbers or fractions)

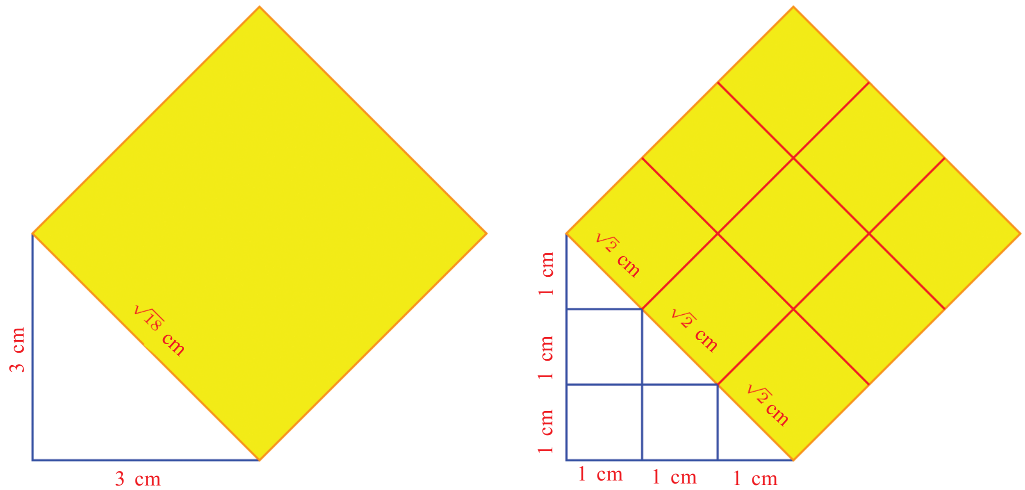
$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$, for any positive numbers x and y.

We can use this to simplify certain square roots. For example, let's compute the hypotenuse of a right triangle of perpendicular sides 3 centimetres each. By Pythagoras Theorem, the area of the square on this hypotenuse is $3^2 + 3^2 = 18$ square centimetres. So its length is $\sqrt{18}$ centimetres.

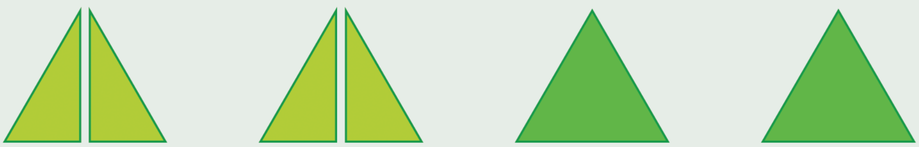
Now writing 18 as 9×2 we get.

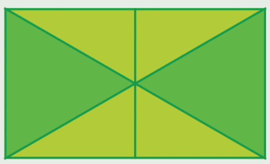
$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$

We can see this geometrically also:



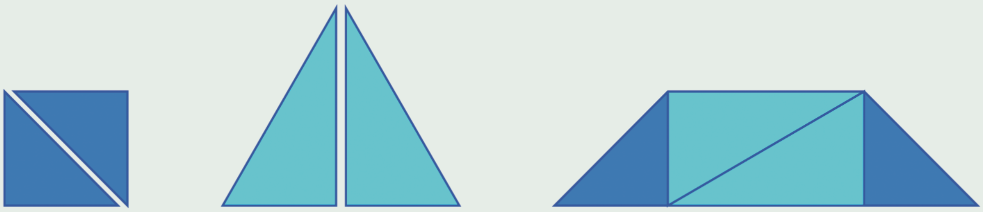
(1) Of four equal equilateral triangles, two are cut vertically into halves and two whole are put together to make a rectangle:





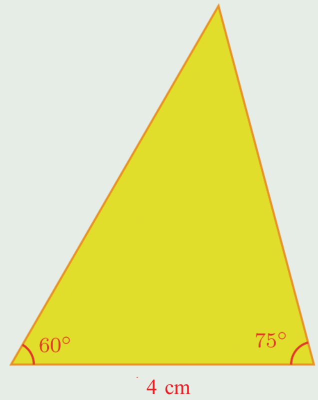
If a side of the triangle is 1 metre, what is the area and the perimeter of the rectangle?

(2) A square and an equilateral triangle of sides twice as long are cut and the pieces are rearranged to form a trapezium, as shown below:



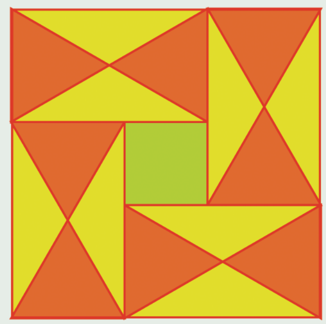
If a side of the square is 2 centimetres, what are the perimeter and area of the trapezium?

(3) Calculate the perimeter and area of the triangle in the picture.



(4) All red triangles in the picture are equilateral.

What is the ratio of the sides of the outer and inner squares?



(5) From the pairs of numbers given below, pick out those whose product is a natural number or a fraction.

i) $\sqrt{3}, \sqrt{12}$ ii) $\sqrt{3}, \sqrt{1.2}$ iii) $\sqrt{5}, \sqrt{8}$

iv) $\sqrt{0.5}, \sqrt{8}$ v) $\sqrt{7\frac{1}{2}}, \sqrt{3\frac{1}{3}}$

Division

We can write the product $2 \times 3 = 6$ as the division $\frac{6}{2} = 3$ or $\frac{6}{3} = 2$. Similarly, the product $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ can also be written as divisions:

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

In general, for any natural number or fraction x, y, we can write the product as $x \times y = z$ and the division as $\frac{z}{x} = y$ and $\frac{z}{y} = x$.

Similarly

For any positive numbers x, y the product

$$\sqrt{x} \times \sqrt{y} = \sqrt{z}$$

can be written as the divisions

$$\frac{\sqrt{z}}{\sqrt{x}} = \sqrt{y} \qquad \frac{\sqrt{z}}{\sqrt{y}} = \sqrt{x}$$

Now since $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$, we have

$$\sqrt{\frac{6}{2}} = \sqrt{3} \qquad \sqrt{\frac{6}{3}} = \sqrt{2}$$

And we have seen earlier that

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

From these pairs of equations, we get

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}}$$

Similarly, from $3 \times \frac{2}{3} = 2$, we get

$$\sqrt{3} \times \sqrt{\frac{2}{3}} = \sqrt{3 \times \frac{2}{3}} = \sqrt{2}$$

and then we can write this product as the division

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

Now let's see how such square roots are computed.

For example, to compute $\sqrt{\frac{1}{2}}$, we first write

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Then we can take a fraction approximately equal to $\sqrt{2}$, divide 1 by this number and thus get a fraction approximately equal to $\frac{1}{\sqrt{2}}$.

$$\frac{1}{\sqrt{2}} \approx \frac{1}{1.414} = 0.707$$ (You can use a calculator).

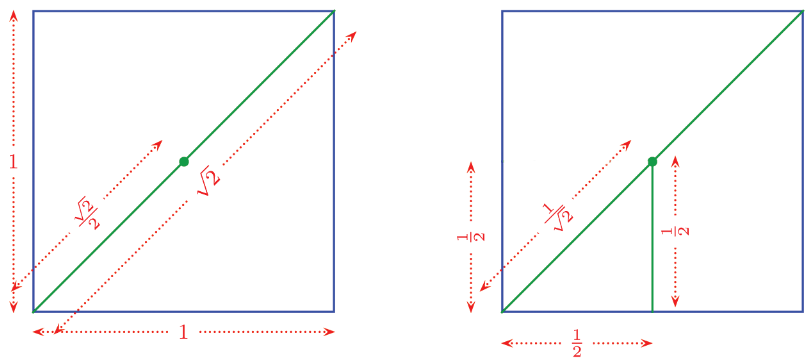
There is an easier way. Since $\frac{1}{2} = \frac{2}{4}$ we can proceed like this:

$$\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Now we easily get

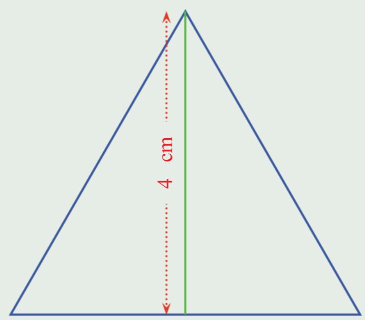
$$\frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707$$ (You don't need a calculator for this!)

We can see $\frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}}$, geometrically also:



Can you compute $\sqrt{\frac{1}{3}}$ like this?

(1) Calculate the length of the sides of the equilateral triangle on the right, correct to a millimetre.

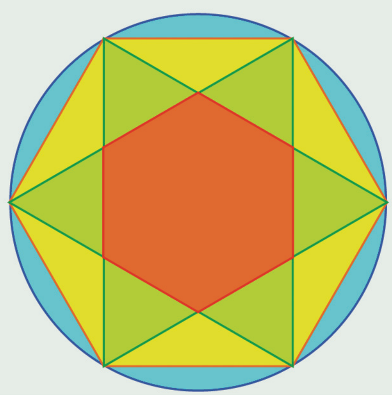


(2) The picture shows the vertices of a regular hexagon connected by lines.

i) Prove that the inner red hexagon is also regular.

ii) How much of a side of the large hexagon is a side of the small hexagon?

iii) How much of the area of the large hexagon is the area of the small hexagon?

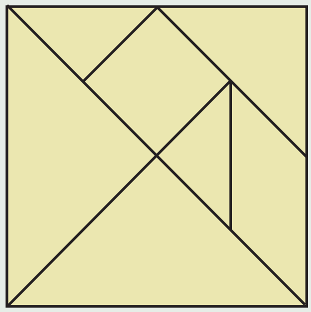


(3) Prove that $(\sqrt{2}+1)(\sqrt{2}-1)=1$. Use this to compute $\frac{1}{\sqrt{2}-1}$ correct to two decimal places.

(4) Compute $\frac{1}{\sqrt{2}+1}$ correct to two decimal places.

(5) Prove that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$ and $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$. Can you find other numbers like this?

(6) The picture shows a tangram of 7 pieces made by cutting a square of side 4 centimetres. Calculate the length of the sides of each piece.



Looking back

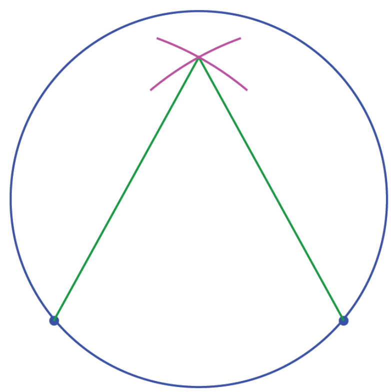
Learning outcomes	On my own	With teacher's help	Must improve
• Justifying the fact that not all lengths can be expressed as fractions. • Explaining the method of expressing as decimals, those square roots which cannot be expressed as fractions. • Explaining the geometric meaning of such numbers and the method to compute the decimal forms of fractions approximately equal to them.			

Circles

Circles and lines

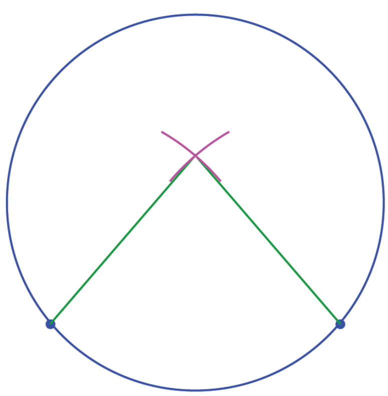
Use a bangle or a small lid to draw a circle in your notebook. How do we find its centre?

The distance from any point on the circle to the centre is the same.



So, if we mark two points on the circle, the centre is a point at the same distance from both. How do you find such a point?

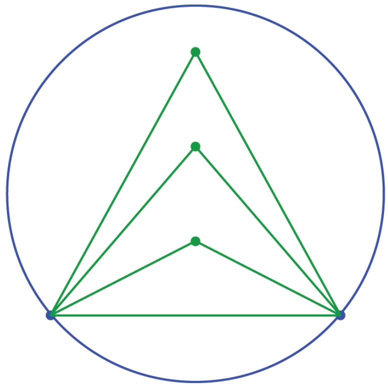
In this picture, the point is above the centre.



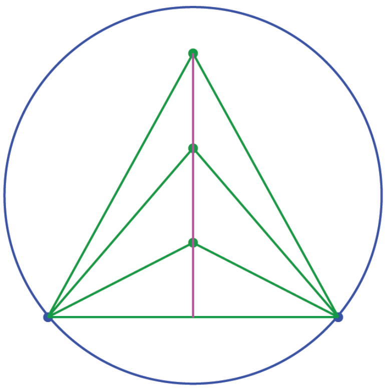
It is not right in this picture also. Instead of proceeding by trial and error like this, let's stop and think a little bit about the problem.

There are so many points at the same distance from the two points marked on the circle. How do we decide which among them is the centre?

All the points which are at the same distance from two fixed points are the third vertices of isosceles triangles with the line joining these two points as base.



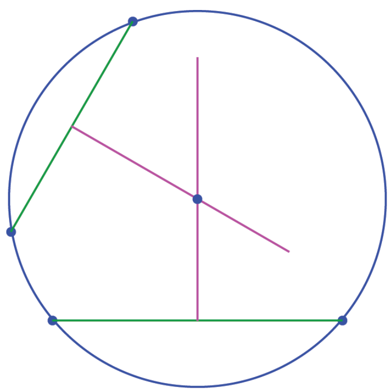
And we know that such points lie along the perpendicular bisector of the base (The lesson, **Equal Triangles** in the Class 8 textbook).



So, the centre we seek is on the perpendicular bisector of the line joining the points we have marked on the circle.

But this doesn't tell us where on the line the centre is.

If we mark two other points on the circle, the centre must be on the perpendicular bisector of the line joining these two points also. Since it must be on both lines, it must be their point of intersection.



The job is done; let's now record what we learnt from it:

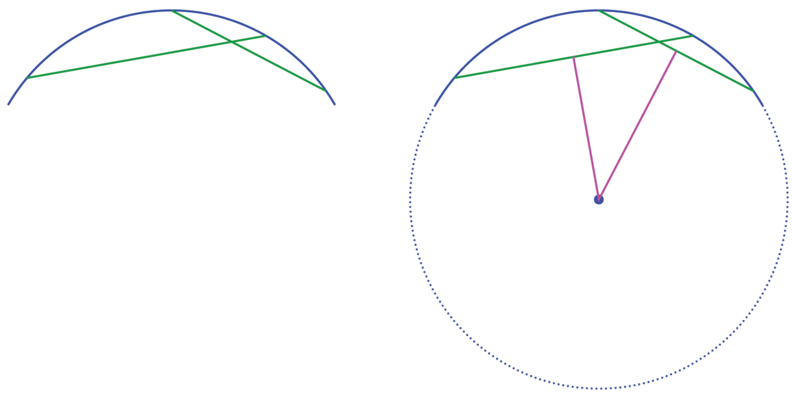
The perpendicular bisector of the line joining any two points on a circle passes through the centre of the circle.

To avoid the long phrase "a line joining two points of a circle", we give such lines a name. (To think a lot and say a little is the way of mathematics). A line joining two points on a circle is called a *chord*.

Thus our general conclusion can be put like this:

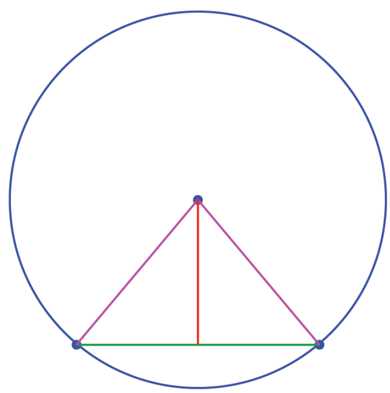
The perpendicular bisector of any chord of a circle passes through its centre.

Now if we just have a part of a circle (a piece of bangle, for example) can't we find the centre and thereby complete the full circle? Just draw two lines inside the piece and draw their perpendicular bisectors:



We reached the above conclusion, starting from the observation that the ends of a chord and the centre form an isosceles triangle. We have seen in Class 8 that the relations between the base and the third vertex of an isosceles triangle can be put in different ways:

- The perpendicular from the third vertex bisects the base.
- The line joining the third vertex and the midpoint of the base is perpendicular to the base.
- The third vertex is on the perpendicular bisector of the base.

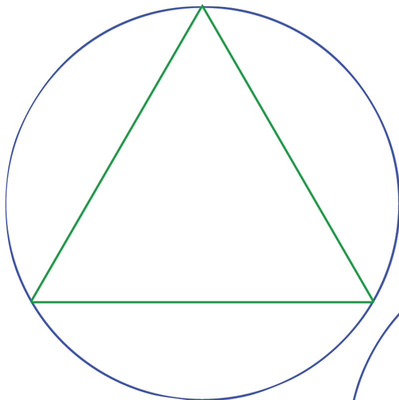


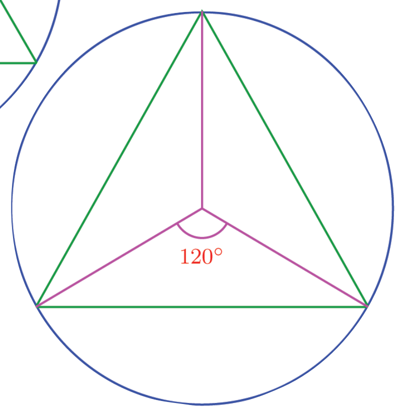
Our statement on chords of a circle is formed by taking the base as a chord and the third vertex as the centre in the third statement above. Similarly we can rewrite the first two statements on triangles as statements about circles.

The perpendicular from the centre of a circle to a chord bisects the chord.
The line joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.

Let’s look at another problem: We need to draw an equilateral triangle with vertices on a given circle. If we draw just any two chords of the same length, the third side may not be equal to these. (Try it!)

So, the first chord itself must be drawn with some care and computation. Let's see how it will look once the triangle is drawn. (We can use GeoGebra for this: draw an equilateral triangle using **Regular Polygon** and use **Circle Through Three Points** to draw a circle passing through its vertices)

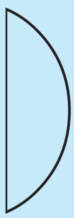




Chord and Cord

A chord of a circle is called *jya* in Sanskrit. This word actually means bowstring.

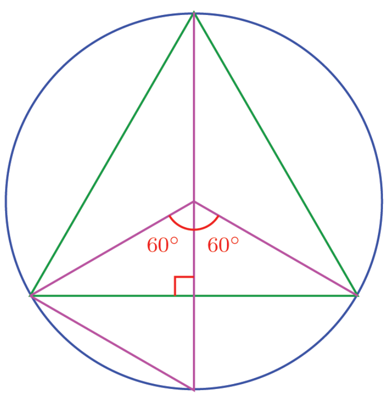
The portion of a circle consisting of a chord and part of the circle connecting its end points does look like a bow, right?

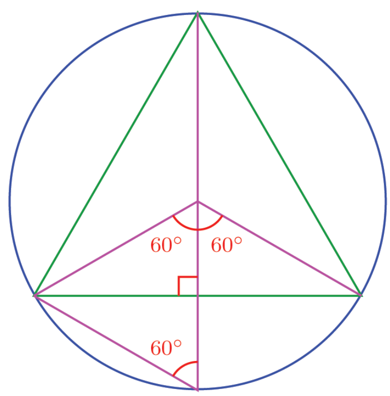


The word chord in English comes from the word *chorda* in Latin, meaning rope. Now we use the word cord in English for a string.

The three triangles obtained by joining the centre of the circle to the vertices are equal. (Why?) So the angles between these lines are all 120°.

Now draw a radius perpendicular to one of the sides:





Can you see how angles in the picture on the left and then in the picture on the right are calculated as 60°?

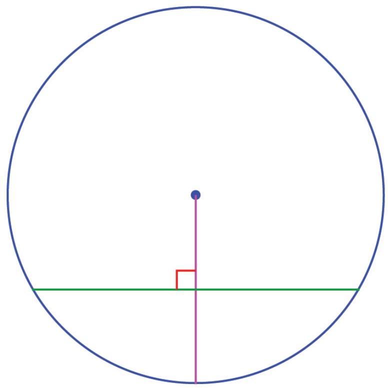
Anyway, the bottom side of the large (green) equilateral triangle is the perpendicular from one vertex of the small (pink) equilateral triangle to its opposite vertex; so it bisects this side of the small triangle.

In other words, a side of an equilateral triangle drawn with vertices on a circle bisects the radius perpendicular to it.

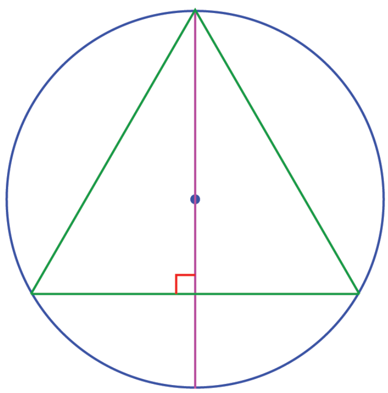
Now there is a question: is it true the other way round?

That is, if a chord which is a perpendicular bisector of some radius is drawn, will it be a side of an equilateral triangle with vertices on the circle?

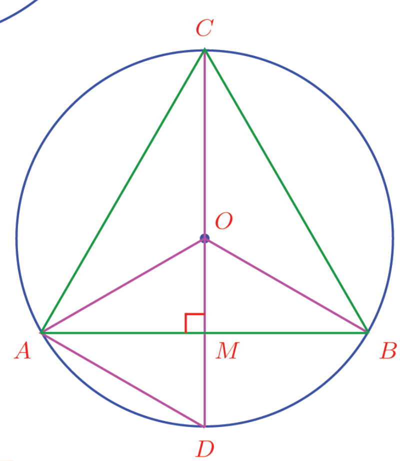
Let's see. Draw a radius and its perpendicular bisector.



Next extend the radius to diameter and join its further end to the ends of the chord.

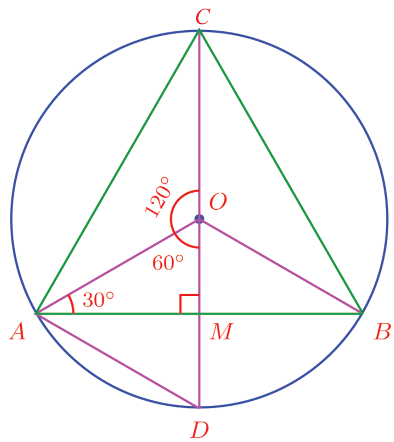


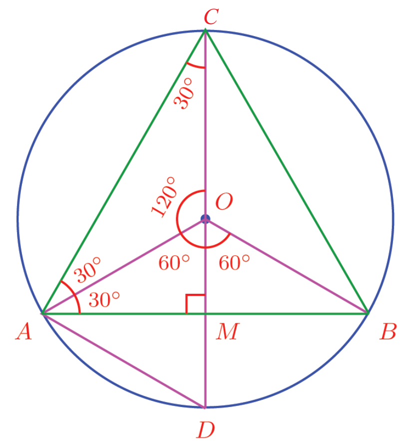
To prove that the triangle obtained thus is equilateral, draw the lines as in this picture.



To prove that ΔABC is equilateral, we need only see that the sides AB and AC are equal and that the angle CAB between them is 60°.

First look at ΔOAD. The sides OA and OD are equal, being radii of the circle. Since the point A is on the perpendicular bisector of OD, the lengths OA and DA are also equal. Thus ΔOAD is equilateral. Using this we can calculate some angles as in the picture on the left; and then some more angles as in the picture on the right:





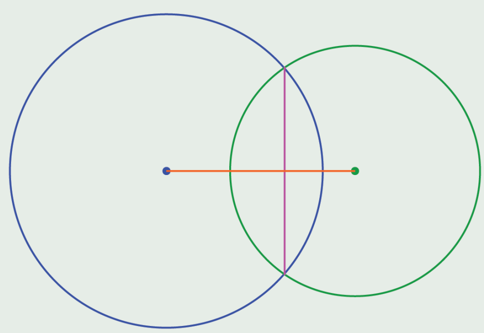
Thus we find that in ΔABC, the angle at A is 60°. To see that AB and AC are equal, look at ΔOAB and ΔOAC. Both have OA as a side. The sides OB and OC are equal. The angle between OA and OB in ΔOAB and the angle between OA and OC in ΔOAC are both 120°. So the sides AB, AC of these triangles are also equal.

Thus ABC is an isosceles triangle with one angle size 60° and so is equilateral.

This is a quick method to draw an equilateral triangle with vertices on a circle, isn't it?

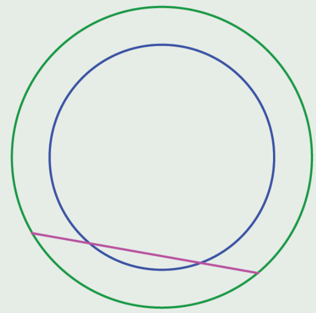
?

(1) Prove that the line joining the centres of two intersecting circles is the perpendicular bisector of the line joining the points of intersection.

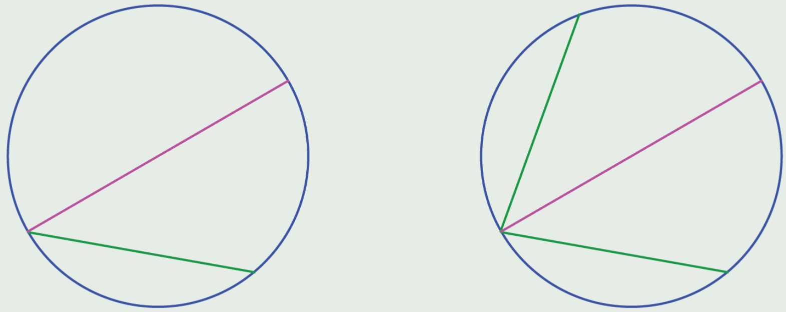


(2) The picture on the right shows two circles centred on the same point and a line intersecting them.

Prove that the parts of the line between the circles on either side are equal.



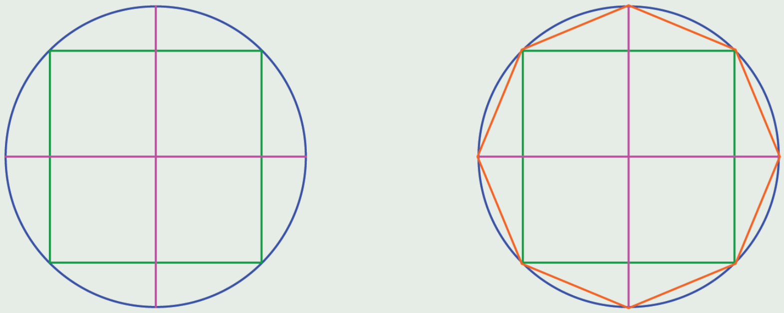
(3) A chord and the diameter through one of its ends are drawn in a circle. A chord of the same inclination is drawn on the other side of the diameter.



Prove that the chords are of the same length.

(4) Prove that the angle made by two equal chords drawn from a point on the circle is bisected by the diameter through that point.

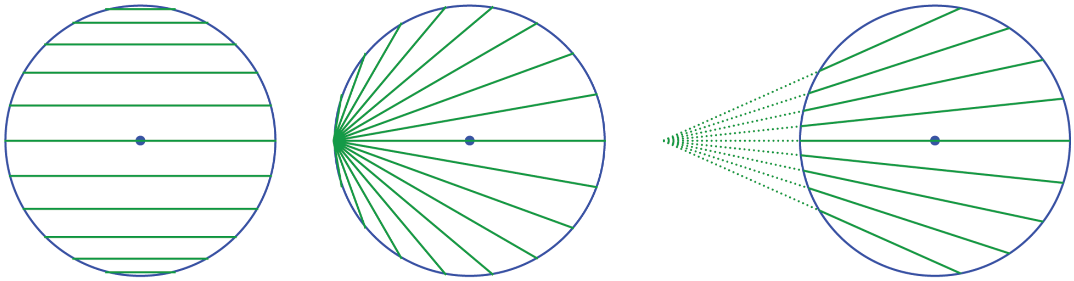
(5) Draw a square and a circle through all four vertices. Draw diameters parallel to the sides of the square and draw a polygon joining the end points of these diameters and the vertices of the square:



Prove that this polygon is a regular octagon.

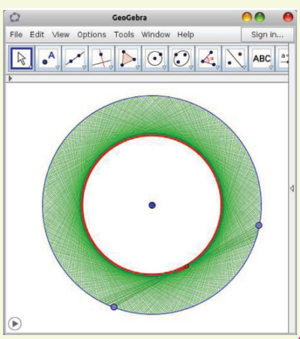
Equal chords

Diameters of a circle are chords through the centre of the circle; and they are also the longest chords. As the chords move away from the centre, their lengths decrease:

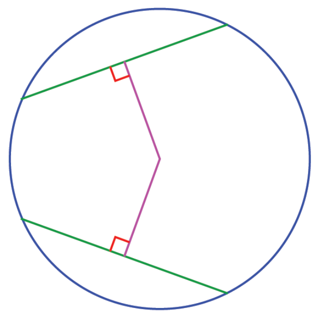


Can you see that the chords at the same distance from the centre are of the same length, whatever way they are moved, by sliding or rotating?

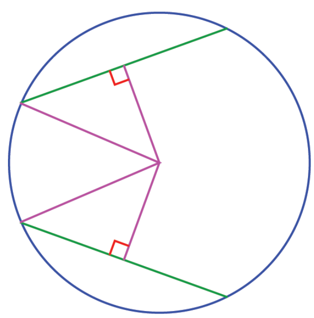
Draw a circle in GeoGebra and join two points on it to draw a chord. Mark its midpoint and enable **Trace On**. Also enable **Animation** for the end points of the chord. What is the path of the midpoint of the chord? Why is this so? Enable Trace on for the chord also. You can colour the chord to get a pretty picture.



See this picture:



Two chords at the same perpendicular distance from the centre. To prove that they are of the same length, join one end of each to the centre:



In the two right triangles obtained thus, the hypotenuses are equal, being radii of the circle. And one pair of perpendicular sides are said to be equal. So by Pythagoras Theorem, the third sides are also equal.

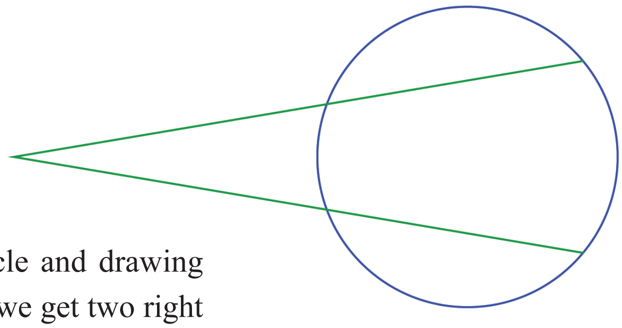
These third sides are half the chords, being the parts cut off by perpendiculars from the centre. Thus we see that half the chords are equal and hence the chords themselves are also equal.

Chords at the same distance from the centre are of the same length.

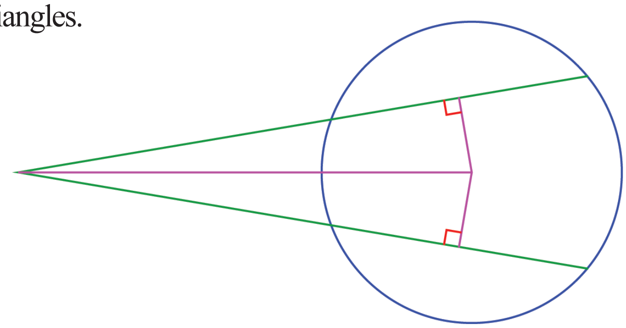
Conversely, starting with equal chords, can you prove that they are at equal distances from the centre? Try it!

Let's look at a problem based on this.

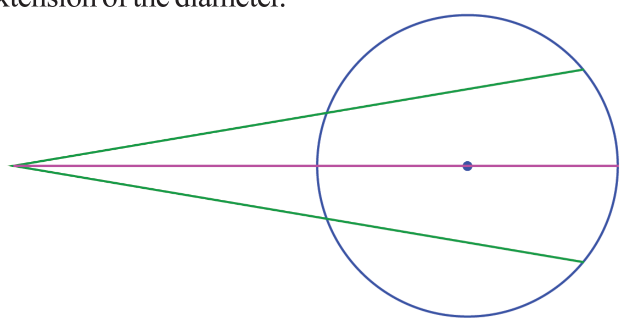
The picture on the right shows two equal chords extended to meet at a point outside the circle.



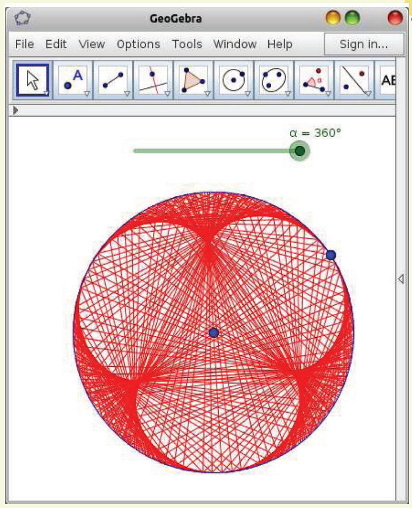
Joining this point with the centre of the circle and drawing perpendiculars from the centre to the chords, we get two right triangles.



They have the same hypotenuse; and since the chords are equal, so are the perpendiculars from the centre. Thus a pair of perpendicular sides of the triangle are also equal. So their angles outside the circle are also equal. That is, the line joining the centre and the point of intersection of the chords is the bisector of the angle between the extended chords. And this line is an extension of the diameter.



We have seen in an earlier problem that if two equal chords meet at a point on the circle, then the diameter through this point bisects the angle between the chords. Now we see that this is true, even if the chords intersect outside the circle.



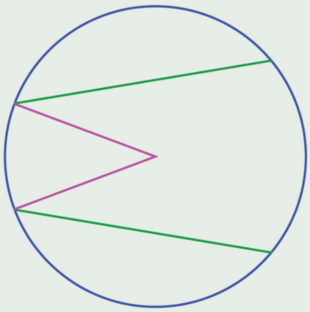
Let's see how a picture like this can be drawn in GeoGebra. Draw a circle centred at a point A and mark a point B on it. Make an angle slider α. Select **Angle with Given Size** and click on B and A in order. In the window coming up, give the size of the angle as α. We get a new point B′. Similarly get another point B″ such that ∠B′AB″ = α. Join B′, B″ and enable **Trace On**. Animate the slider and see. Instead of specifying ∠B′AB″ as α, try 2α, 3α, 4α, ... The picture shows what we get for ∠B′AB″ = 3α.

(1) Prove that chords of the same length in a circle are at the same distance from the centre.

(2) Two chords intersect at a point on a circle and the diameter through this point bisects the angle between the chords. Prove that the chords have the same length.

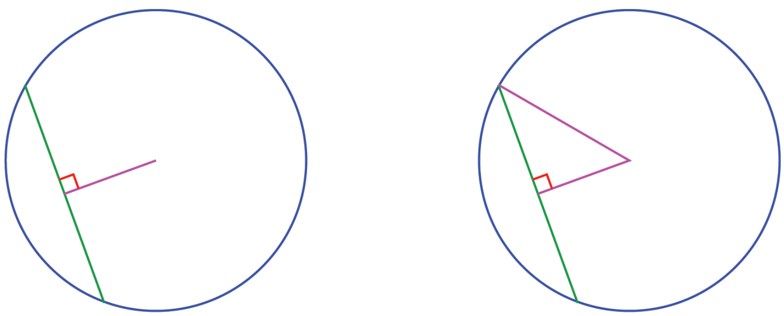
(3) In the picture on the right, the angles between the radii and the chords are equal.

Prove that the chords are of the same length.



Length of Chords

We have seen that length of a chord is determined by the distance from the centre. Let's look at the actual computation now.



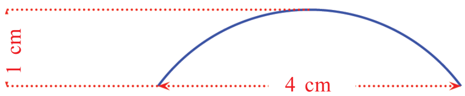
The picture on the left shows a chord of a circle and the perpendicular from the centre. In the picture on the right, one end of the chord is joined to the centre, to form a right triangle.

The hypotenuse of this right triangle is the radius of the circle, one of the shorter sides is the perpendicular from the centre and the third side is half the chord. So, we can calculate the length of the chord using Pythagoras Theorem:

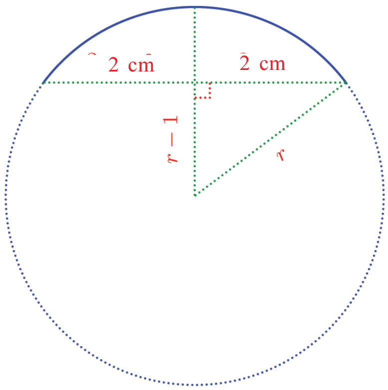
In a circle, the square of half a chord is the difference of the squares of the radius and the perpendicular from the centre to the chord.

For example, in a circle of radius 4 centimetres, the square of half a chord at a (perpendicular) distance 3 centimetres from the centre is $4^2 - 3^2 = 7$; so the length of the chords is $2\sqrt{7}$ centimetres.

Now consider this problem: The distance between the ends of a piece of a bangle is 4 centimetres and its height is 1 centimetre:



We have to calculate the radius of the full bangle. We can imagine the bangle like this:



Taking the radius of the bangle to be r, we have from the right triangle in the picture,

$$r^2 - (r-1)^2 = 4$$

Simplifying this, we get $2r - 1 = 4$ and so $r = 2\frac{1}{2}$. Thus the radius of the bangle is 2.5 centimetres.

Lotus problem

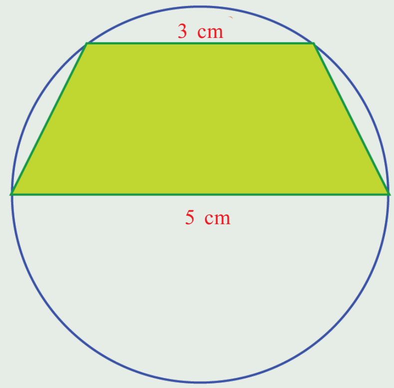
Haven't you heard about the mathematical text *Leelavati* by Bhaskaracharya? Here is the translation of one of its verses:

"In a lake full of frolicking birds,
Stands a lotus bud
Half a palm high
Moving lazily in the wind
It sank down at two palms
Tell me quickly, mathematician
How deep the water is!"

(1) In a circle, a chord 1 centimetre away from the centre is 6 centimetres long. What is the length of a chord 2 centimetres away from the centre?

(2) In a circle of radius 5 centimetres, two parallel chords of lengths 6 and 8 centimetres are drawn on either side of a diameter. What is the distance between them? If parallel chords of these lengths are drawn on the same side of a diameter, what would be the distance between them?

(3) The bottom side of the quadrilateral in the picture is a diameter of the circle and the top side is a chord parallel to it. Calculate the area of the quadrilateral.



(4) In a circle, two parallel chords of lengths 4 and 6 centimetres are 5 centimetres apart. What is the radius of the circle?

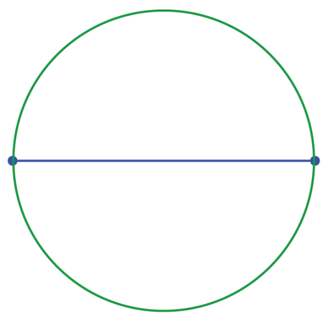
Points and circles

We have been talking about lines joining two points on a circle. Now consider a question in reverse: How do we draw a circle through the ends of a line?

We can join any two points by a line. So the question can be put like this: Can we draw a circle through any two points?

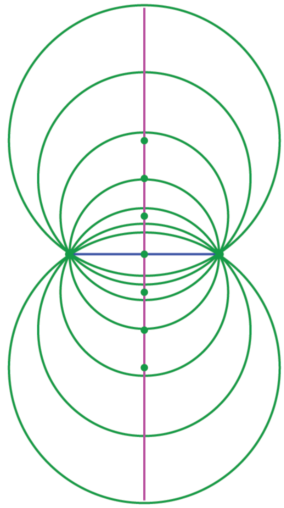
Mark two points in your notebook. Can you draw a circle passing through them?

A quick solution is to draw a circle with the line joining the points as diameter. Can you draw another circle?



If such a circle is drawn, the line joining the points would be a chord. So, the centre of the circle would be on its perpendicular bisector.

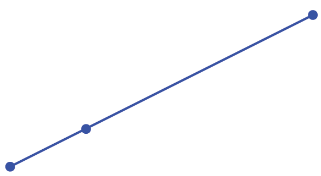
On the other hand, we can choose any point on this bisector as the centre, to draw a circle through these points, right?



Draw a line and its perpendicular bisector in GeoGebra. Mark a point on the bisector and draw a circle with centre at this point and passing through an end point of the line. Enable **Animation** for the centre. **Trace On** may be enabled for the circle.

Now a new question, can we draw a circle through any three points?

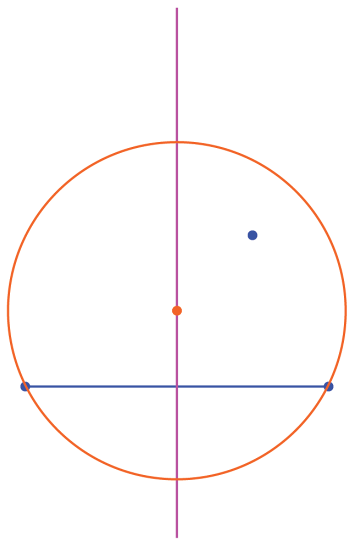
If the points are on a line, we cannot.



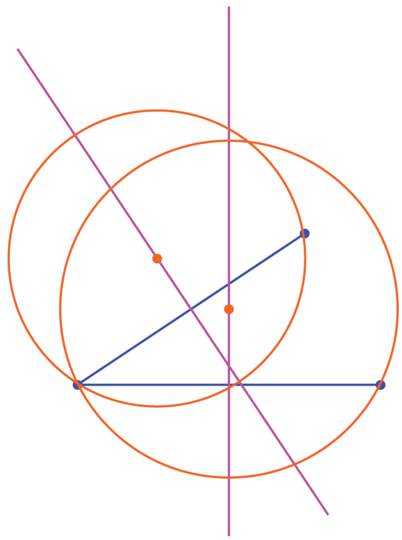
What if they are not on a line?

Let's think a bit, before we try to do it.

We can draw a circle through any two of the points given by choosing a centre and a point on the perpendicular bisector of the line joining them.



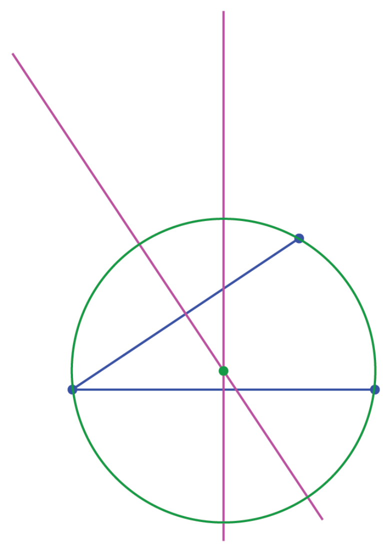
Taking another pair of points and choosing a point on the perpendicular bisector of the line joining them as centre, we can draw a circle passing through them.



Thus we can draw two circles passing through two pairs of points. But what we need is a single circle passing through all three points.

For a circle through the first pair of points, the centre must be on the first bisector and for a circle through the second pair, the centre must be on the second bisector.

What if we take as the centre, a point on both the bisectors, that is, their point of intersection?



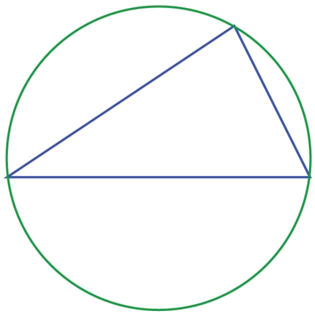
Line and circle

Through one point, we can draw as many lines as we want; and circles too.

Through two points, we can draw only one line; but as many circles as we want.

It may not be possible to draw a line through three points. If we can draw a line through three points, we cannot draw a circle through them; and if the three points are such that we cannot draw a line through them, then we can draw a circle through them.

If we join the remaining pair of points also, we get a triangle; and the circle passes through all its vertices.

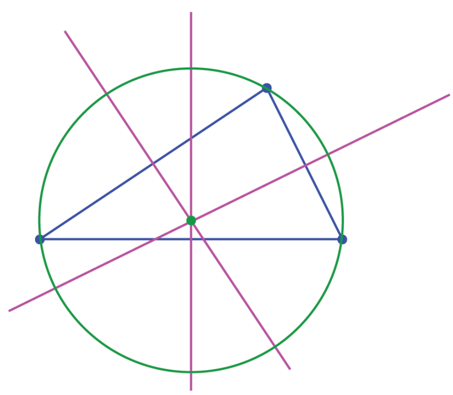


Such a circle, passing through all three vertices of a triangle is called the circumcircle of the triangle.

We can draw the circumcircle of any triangle by choosing as centre, the point of intersection of the perpendicular bisectors of two sides, as we have done just now.

In GeoGebra, we can draw a Circle Through 3 Points by using a tool of this name. Make an angle Slider α and draw a triangle with one angle α. Draw its circumcircle and mark its centre using **Midpoint or Centre.** Move the slider to change α and see how the position of circumcentre changes. When is it inside the triangle? When is it outside? Will it be on a side of the triangle at any time?

We can note another thing here. In our example, we drew the perpendicular bisectors of the bottom and left sides of the triangle to get the centre of the circumcircle. Since the right side is also a chord of the circumcircle, its perpendicular bisector also passes through the centre.



In any triangle, the perpendicular bisectors of all three sides intersect at a single point.

(1) Draw three triangles with lengths of two sides 4 and 5 centimetres and the angle between them 60°, 90°, 120°. Draw the circumcircle of each. (Note how the position of the circumcentre changes).

(2) The equal sides of an isosceles triangle are 8 centimetres long and the radius of its circumcircle is 5 centimetres. Calculate the length of its third side.

(3) Find the relation between a side and the circumradius of an equilateral triangle.

Looking back

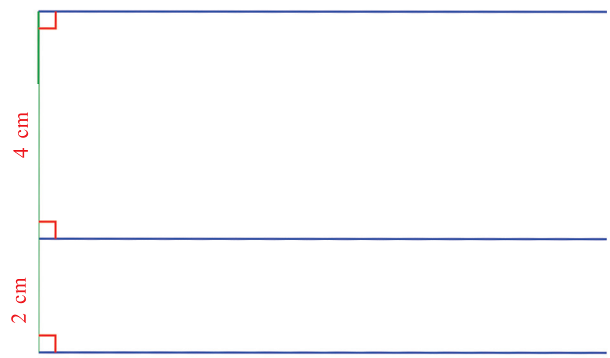
Learning outcomes	On my own	With teacher's help	Must improve
• Explaining the relation between a chord of a circle and the perpendicular from the centre in various ways. • Drawing a full circle from a part of it. • Explaining the relations between equal chords in terms of the distance from the centre and in terms of the angles made with the diameter through the point of intersection. • Finding out the relation between the length of chords and the distances from the centre.			

6 Parallel Lines

Parallel division

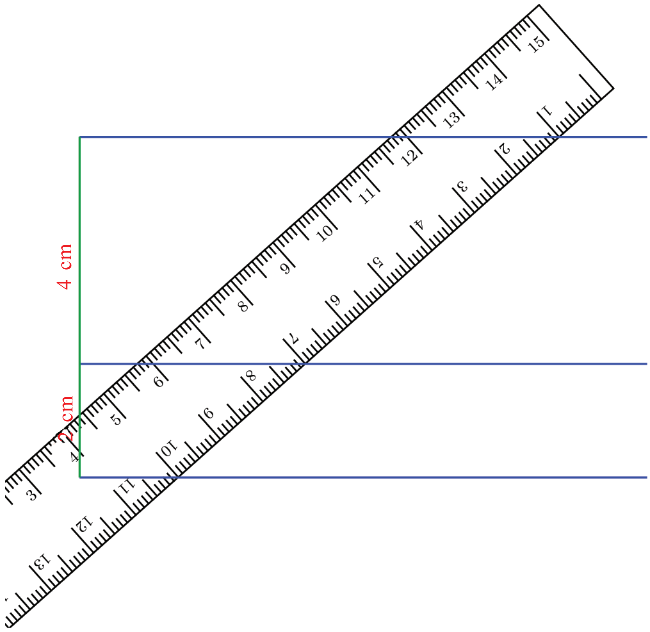
We have learnt many things about parallel lines and have drawn many figures using them. There is much more.

Let's start by drawing a line, another line parallel to it 2 centimetres below and one more line parallel to these, 4 centimetres above:



Now if we measure vertically from any point on the bottom line, the distances are again 2 centimetres and 4 centimetres:

What if we measure at a slant?



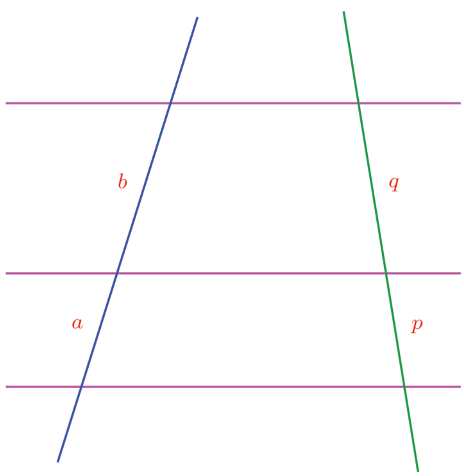
Look at the right edge of the ruler. What are the distances along it?

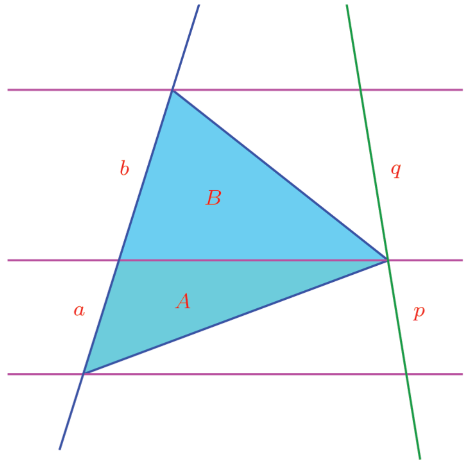
Measure along different slants. What do you see?

Along whatever line we measure, the distance from the middle line to the top line is double the distance from the bottom line to the middle line, isn't it?

In other words, the ratio of the distances along any line is the same as the ratio of the vertical distances, right?

Let's see if this is true for any three parallel lines. First let's make our guess clear. See this picture:





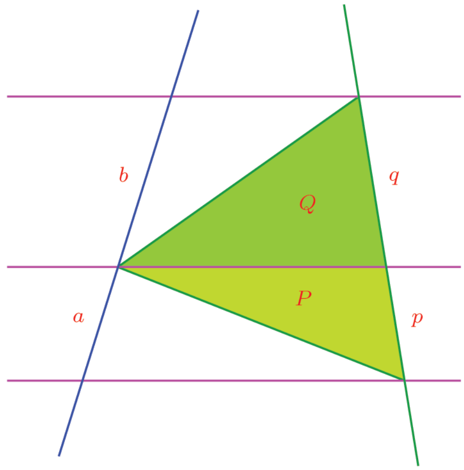
There are three horizontal lines parallel to one another and two slanted lines cutting them across. Taking the lengths of the parts cut off on the left line as a, b and those on the right line as p, q, we have to verify whether the ratios $a : b$ and $p : q$ are the same.

For this, we first convert the ratio $a : b$ of lengths into a ratio of areas.

See the picture on the right. Isn't $a : b$, the ratio of the areas of the lower and upper triangles? (The section, **Triangle division** of the lesson, **Area**)

Taking these areas as A and B, we have

$$\frac{a}{b} = \frac{A}{B}$$

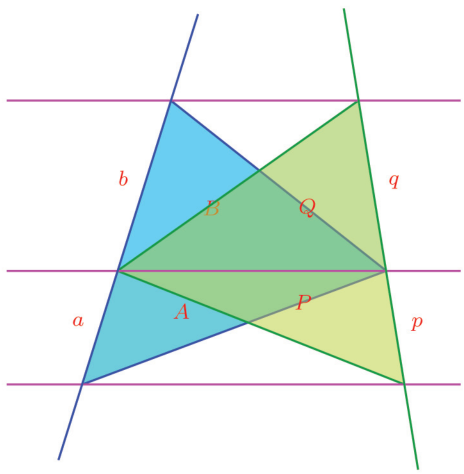


In the same way, the ratio of the lengths p, q can be rewritten as a ratio of areas.

If we take P and Q as the areas of the green triangles, then

$$\frac{p}{q} = \frac{P}{Q}$$

Now let's look at all the triangles together:



Now the lower blue and green triangles share a common side; and their third vertices are on a line parallel to this side. So, they have the same area:

$$A = P$$

Things are the same for the upper blue and green triangles also, aren't they? So,

$$B = Q$$

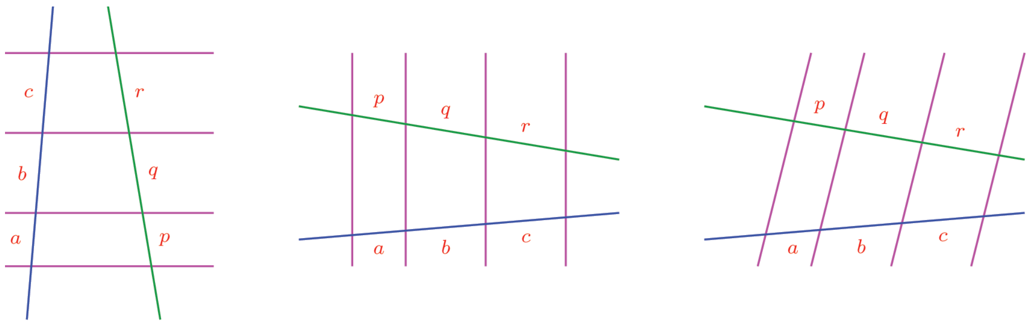
We have seen that $\frac{a}{b} = \frac{A}{B}$ and $\frac{p}{q} = \frac{P}{Q}$. Now we find that $A = P$ and $B = Q$ also. Putting all these together, we get

$$\frac{a}{b} = \frac{p}{q}$$

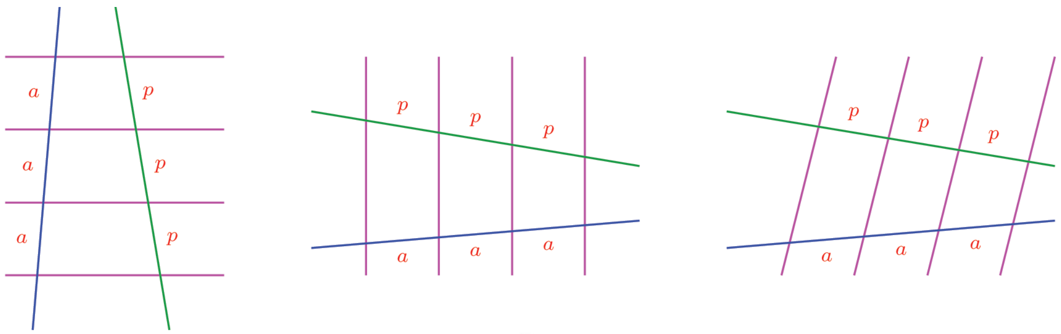
Thus any three parallel lines cut any two lines into pieces whose lengths are in the same ratio. We can extend the above reasoning to more than three parallel lines also.

Three or more parallel lines cut any two lines in the same ratio.

For example, in all three pictures below, the lengths a, b and c are in the same ratio as the lengths p, q and r:



So what if some set of parallel lines cut a line into equal pieces? According to our general principle, they will cut any other line also into equal pieces.



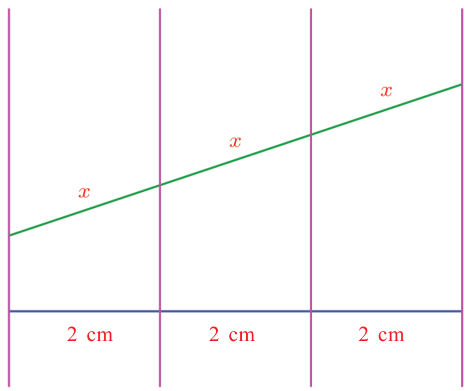
If three or more parallel lines cut a line into equal parts, they will cut any line into equal parts.

Now let's see how these can be applied.

A 7 centimetre long line can be cut into two equal pieces by drawing the perpendicular bisector; or we can just mark the point 3.5 centimetres away from one end. How do we divide it into three equal parts?

It is easy to do in a 6 centimetre long line.

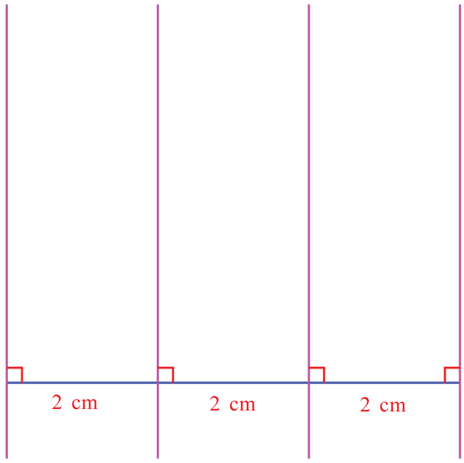
Four parallel lines cutting a 6 centimetre line into three equal parts, will cut any other line into three equal parts, right?

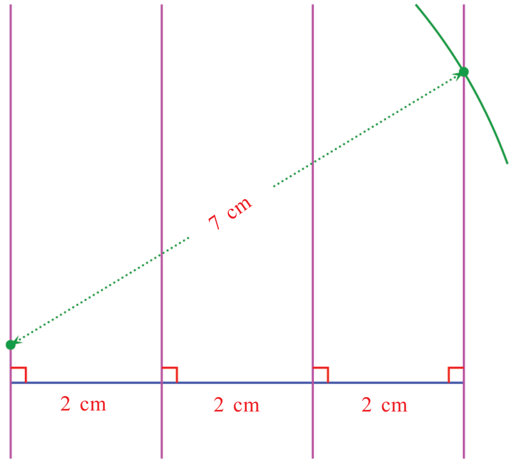


Suppose we make the length of the second line 7 centimetres?

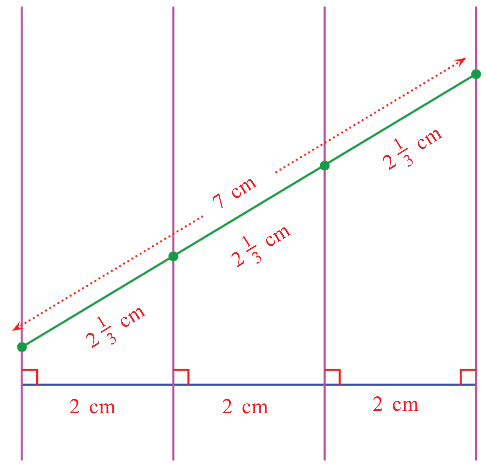
So our way ahead is clear, isn't it?

Draw a line 6 centimetres long and draw perpendiculars to it, 2 centimetres apart. With some point on the first perpendicular as the centre, draw an arc of radius 7 centimetres to cut the last perpendicular.

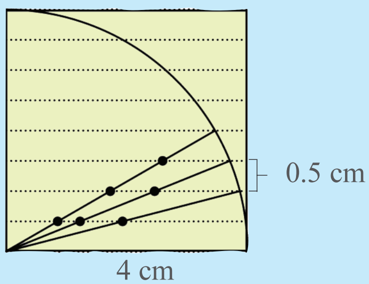




Joining the first and last points, we have a line 7 centimetres long, cut into three equal parts.



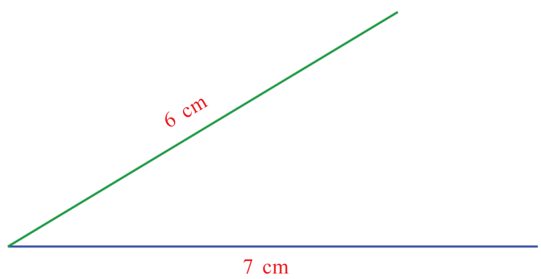
Circle division



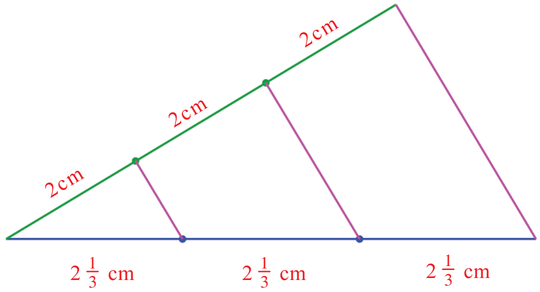
See how a 4 centimetres long line is divided into two, three and four equal parts in this picture. Using this, can't we divide it up to eight equal parts? Like this can you divide a 6 centimetres line into 7 equal parts, using ruled paper?

There is a slightly different way to do this:

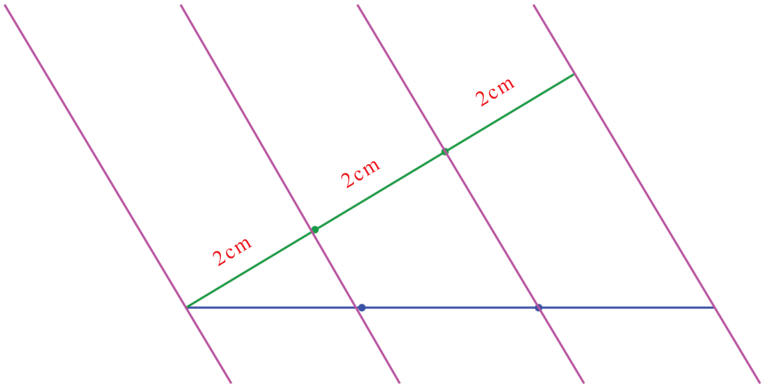
Draw a line 7 centimetres long and from one end, draw another line 6 centimetres long at a slant.



Join the other ends of these lines. Divide the upper line into three equal parts and draw parallel lines through these points.



If you can't see why this works, imagine the parallel lines extended a bit and also a fourth parallel:



Shadow math

The height up to the lowest branch of a tree is 1 metre and the length of the shadow of this part is 2 metres. The total length of the shadow is 8 metres. What is the height of the tree?

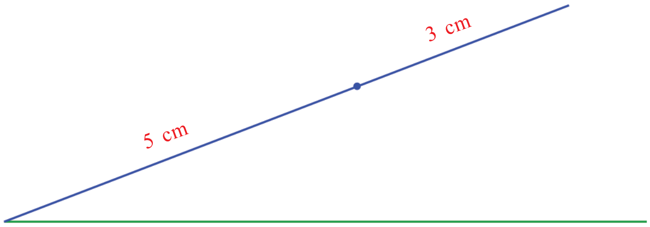
Let's look at another problem.

What is the perimeter of the rectangle on the right?

How do we draw a rectangle with sides in the same ratio and perimeter 17 centimetres?

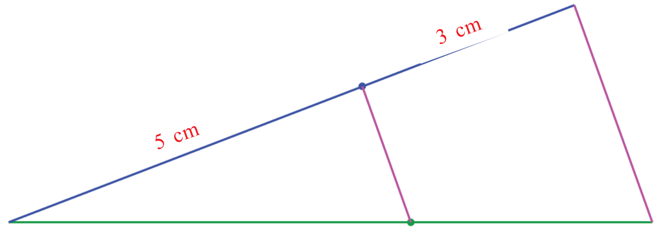
Perimeter of 17 centimetres means the sum of the lengths of the sides is 8.5 centimetres: So we need only draw a line 8.5 centimetres long, divide it in the ratio 5 : 3, and use the parts as the sides of the rectangle.

So, let's first draw a line 8.5 centimetres long. To divide it in the ratio 5 : 3, we proceed as in the second method of doing the first problem. Draw a line 8 centimetres long from one end and divide it into 5 centimetres and 3 centimetres parts:

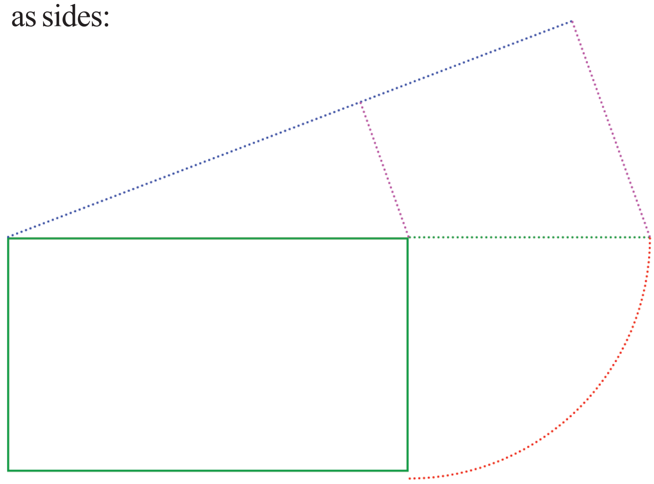


Draw lines AB and AC in GeoGebra. Make a slider c with **Min = 0** and **Max = 1**. With A as centre, draw a circle of radius c times the length of AB. (For this, give the radius of the circle as c * AB or ca. Here a is the length of AB). Mark the point D where this circle cuts AB. Similarly, mark the point E on AC at a distance c times the length of AC, from A. Draw the lines AD, AE and mark their lengths. What is the relation between them? Move the slider and see. Draw the lines BC and DE. what is the relation between them?

Now join the other ends of the lines and draw a line parallel to it through the point of division of the shorter lines to cut the longer line in the ratio 5 : 3.



And the rectangle we require is the one with these parts as sides:

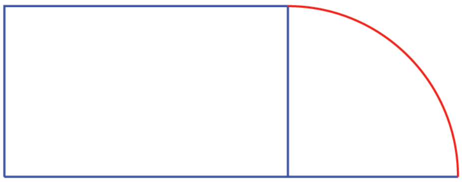


A slightly different problem:

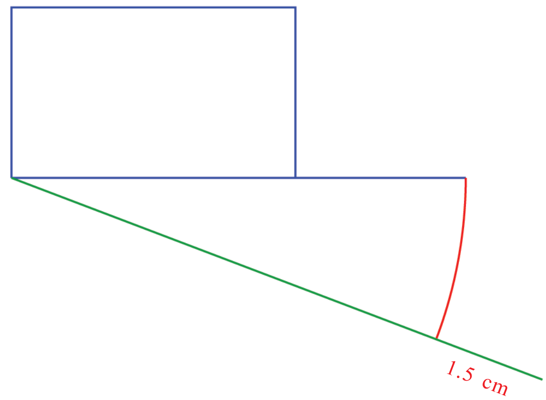
A rectangle is shown here.

Its length and breadth are not given. We must draw a rectangle with the same ratio of sides, but with perimeter 3 centimetres more.

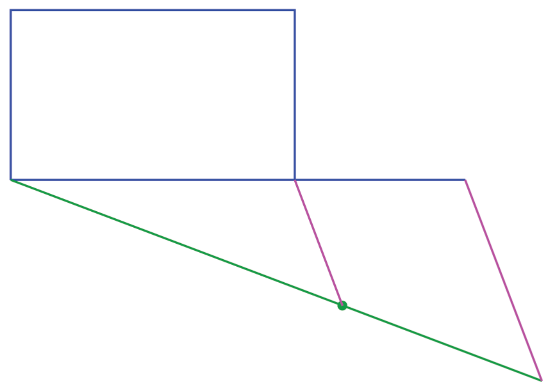
For this, we first put the length and width of the given rectangle along one line:



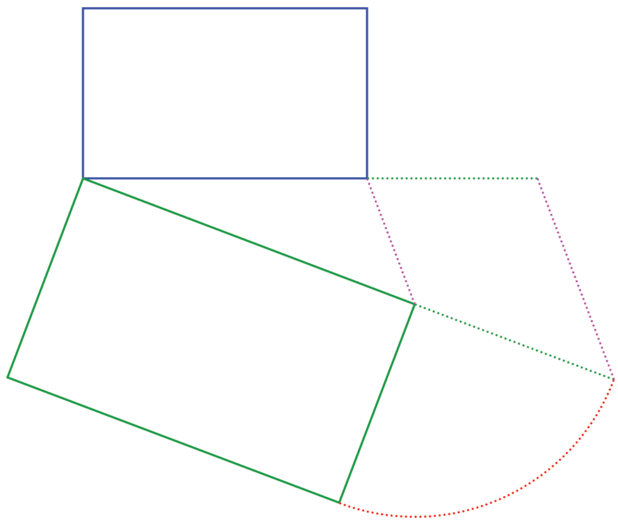
Now we draw below it another line of the same length at a slant and extend it by 1.5 centimetres. (Why?)

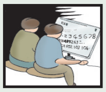


Now join the other ends of the lines and draw a parallel to it to cut the lower line.



Now we can draw the required rectangle with parts of the lower line as sides.





(1) Draw an 8 centimetres long line and divide it in the ratio 2 : 3.

(2) Draw a rectangle of perimeter 15 centimetres and sides in the ratio 3 : 4.

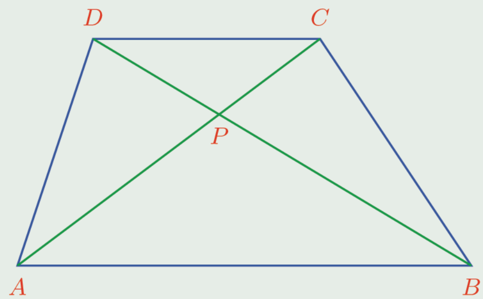
(3) Draw triangles specified below, each of perimeter 10 centimetres.

i) Equilateral triangle

ii) Sides in the ratio 3 : 4 : 5

iii) Sides in the ratio 2 : 3 : 4

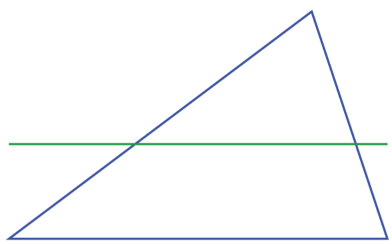
(4) In the picture below, the diagonals of the trapezium *ABCD* intersect at *P*.

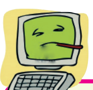


Prove that $PA \times PD = PB \times PC$

Triangle division

Draw a triangle and within it, draw a line parallel to one side.

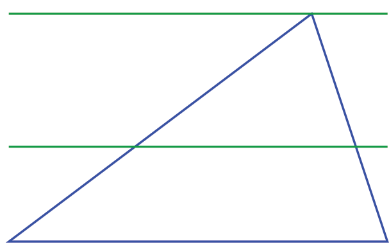




Draw triangle ABC in GeoGebra and mark a point D on AB. Draw a line through D, parallel to BC and mark the point E where it cuts AC. Check if D and E divide AB and AC in the same ratio. You can mark the lengths and see this.

Is there any relation between the parts into which the line divides the other sides?

What if we draw another line parallel to the bottom side, through the top vertex?



Now three parallel lines cut the left and right sides of the triangle. The ratio of the parts must be the same. And these parts are those cut by the first line.

So, what do we see here?

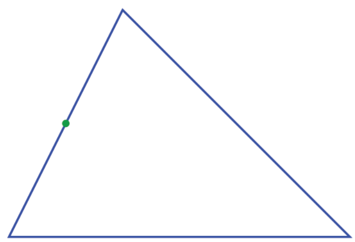
In any triangle, a line drawn parallel to a side cuts the other two sides in the same ratio.

What if the parallel line is drawn through the mid point of a side?

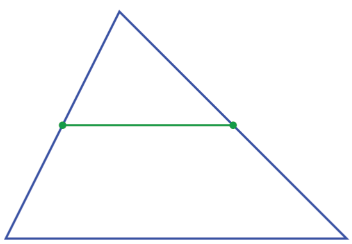
The parts cut on that side are equal; so by the general principle, the parts of the other side must also be equal. This also is something worth noting.

In any triangle, the line drawn parallel to one side, passing through the midpoint of another side, meets the third side also at its midpoint.

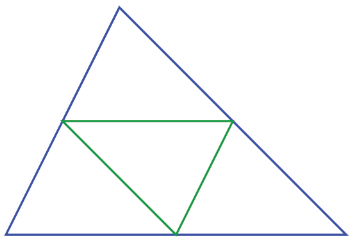
Now see this triangle:

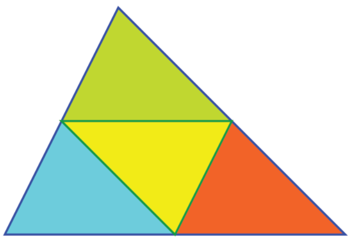


We want to draw a line through the midpoint of the left side, parallel to the bottom side. We have noted that this line passes through the mid point of the right side also. So, joining the mid points of the left and right sides, we get the required line parallel to the bottom side.



What if we connect the mid points of all three sides?





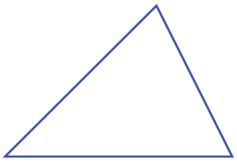
What can we say about these four smaller triangles? The sides of the yellow triangle in the middle are parallel to the sides of the large triangle.

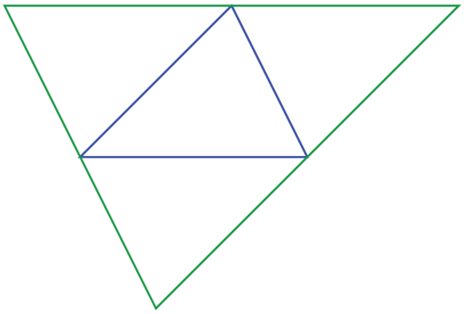
Don't all four triangles look equal? Let's check whether it is true. Let's take the yellow and blue triangles. The left side of the yellow triangle is the same as the right side of the blue triangle. The lower angle on this side in the yellow triangle is equal to the upper angle on this side in the blue triangle. (Why?)

In the same way, the other angles on this side in the two triangles are also equal. So these two triangles are equal. Similarly, the red triangle and the green triangle can also be seen to be equal to the yellow triangle. Thus all the four triangles are equal. We note one thing from this: since the sides of these triangles are of the same length, each is half a side of the large triangle.

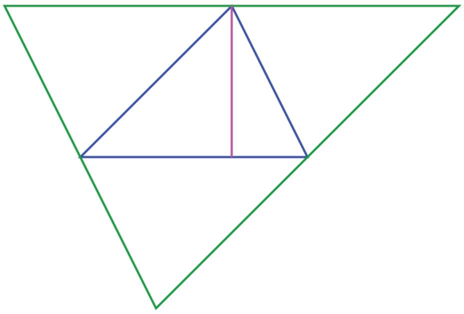
The length of the line joining the midpoints of two sides of a triangle is half the length of the third side.

Now suppose we start with a small triangle and draw through each vertex, the line parallel to the opposite side:





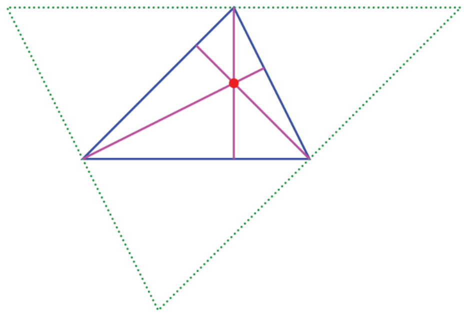
We get a large triangle, made up of three more copies of the small triangle.



Here, the line through a vertex of the small triangle, perpendicular to the opposite side, is the perpendicular bisector of the large triangle.

So, what if we draw all three lines from each vertex of the small triangle, perpendicular to the opposite side? We get the perpendicular bisectors of all three sides of the large triangle.

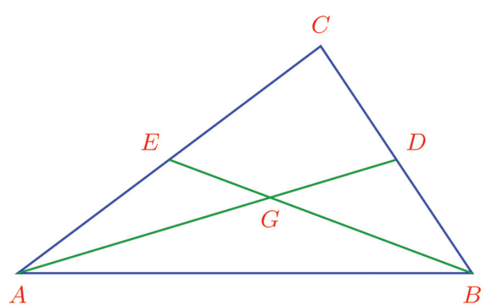
We have seen in the lesson **Circles**, that all the perpendicular bisectors of the sides of any triangle passes through a single point.



In any triangle, all the perpendiculars from the vertices to the opposite sides passes through a single point.

Using the same principle, we can also show that the lines joining the vertices of a triangle to the midpoint of the opposite sides passes through a single point. Such a line is called a *median* of the triangle.

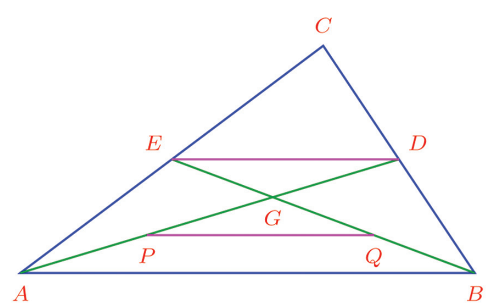
In the picture below, the medians from the two bottom vertices intersect at G.



The line joining the midpoints of the left and right side is parallel to the bottom side and of half the length of this side. That is

$$ED = \frac{1}{2} AB$$

Now there is also a small triangle GAB on the bottom side. Let's join the mid points of the left and right sides of this triangle also.

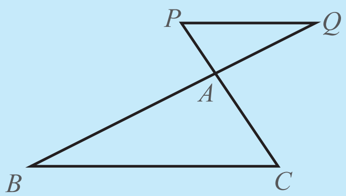


$$PQ = \frac{1}{2} AB$$

External line

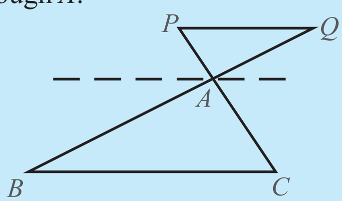
We can show that a line outside a triangle parallel to one side also intersects the other two sides in the same ratio.

See this picture:



PQ is parallel to BC.

Draw another line parallel to BC through A.



So,

$$\frac{AC}{AP} = \frac{AB}{AQ}$$

Also from the picture, we see that

$$\frac{PC}{AP} = \frac{AP + AC}{AP} = 1 + \frac{AC}{AP}$$

$$\frac{QB}{AQ} = \frac{AQ + AB}{AQ} = 1 + \frac{AB}{AQ}$$

From these equations we see that

$$\frac{AP}{PC} = \frac{AQ}{QB}$$

So,

$$PQ = ED$$

Since the sides PQ and ED of the quadrilateral $PQDE$ are equal and parallel, it is a parallelogram. So, its diagonals bisect each other. That is,

$$PG = GD$$

P is the mid point of AG, so that

$$AP = PG = GD$$

Similarly,

$$BQ = QG = GE$$

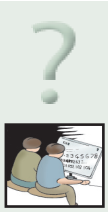
Thus the point of intersection of the two medians divide each other in the ratio 2 : 1.

Now suppose we draw the medians through B and C, instead of those through A and B.

Their point of intersection would divide BE in the ratio 2 : 1. Thus, the point of intersection would be G itself.

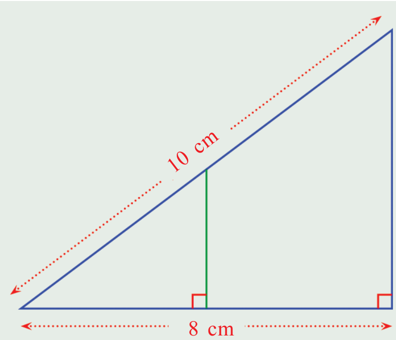
In any triangle, all the medians intersect at a single point; and that point divides each median in the ratio 2 : 1, measured from the vertex.

The point of intersection of the medians is called the *centroid* of the triangle.

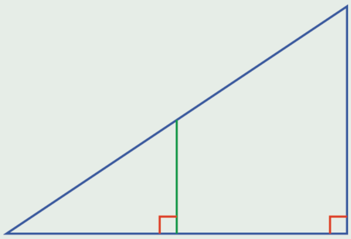


(1) In the picture, the perpendicular is drawn from the midpoint of the hypotenuse of a right triangle to the base.

Calculate the length of the third side of the large right triangle and the lengths of all three sides of the small right triangle.



(2) Draw a right triangle and the perpendicular from the midpoint of the hypotenuse to the base.

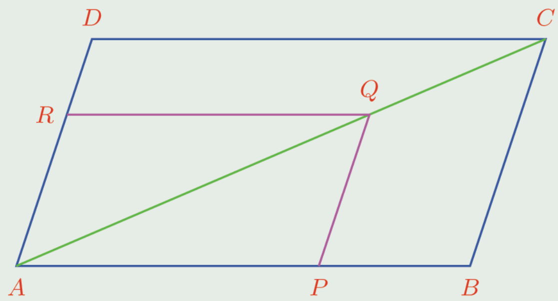


i) Prove that this perpendicular is half the perpendicular side of the large triangle.

ii) Prove that in the large triangle, the distances from the midpoint of the hypotenuse to all the vertices are equal.

iii) Prove that the circumcentre of a right triangle is the midpoint of its hypotenuse.

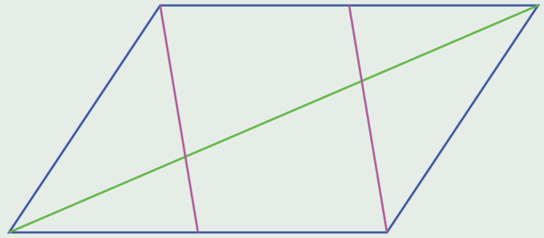
(3) In the parallelogram $ABCD$, the line drawn through a point P on AB, parallel to BC, meets AC at Q. The line through Q, parallel to AB meets AD at R.



i) Prove that $\frac{AP}{PB} = \frac{AR}{RD}$

ii) Prove that $\frac{AP}{AB} = \frac{AR}{AD}$

(4) In the picture below, two vertices of a parallelogram are joined to the midpoints of two sides.

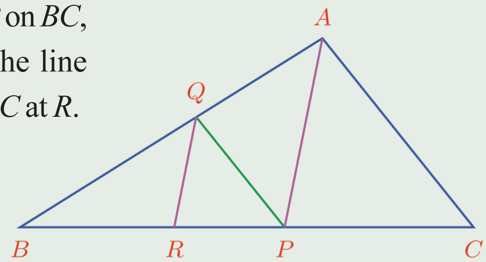


Prove that these lines divide the diagonal in the picture into three equal parts.

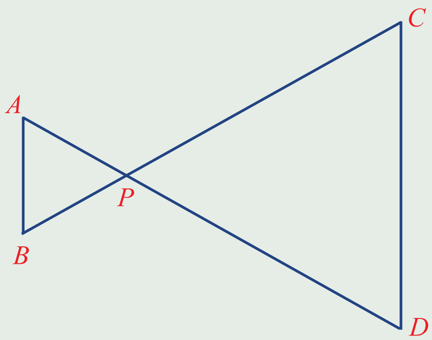
(5) Prove that the quadrilateral formed by joining the mid points of a quadrilateral is a parallelogram. What if the original quadrilateral is a rectangle? What if it is a square?

(6) In ΔABC, the line through a point P on BC, parallel to AC meets AB at Q. The line through Q, parallel to AP, meets BC at R.

Prove that $\frac{BP}{PC} = \frac{BR}{RP}$



(7) AB and CD are parallel lines in the picture.



Prove that $AP \times PC = BP \times PD$

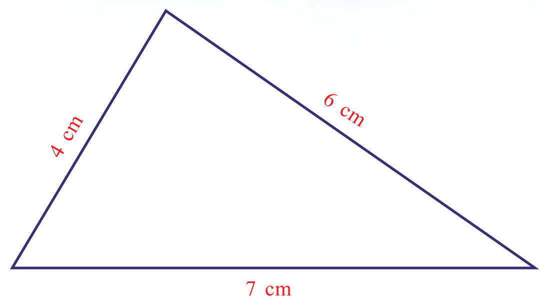
Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Explaining that parallel lines intersect other lines in the same ratio.			
• Dividing a line in a specified ratio using this.			
• Drawing rectangles by increasing or reducing the dimensions without altering the ratio of sides.			
• Explaining the connections between the parts into which a line parallel to a side of a triangle cuts the other sides.			
• Describing the idea of the centroid of a triangle.			

7 Similarity of Triangles

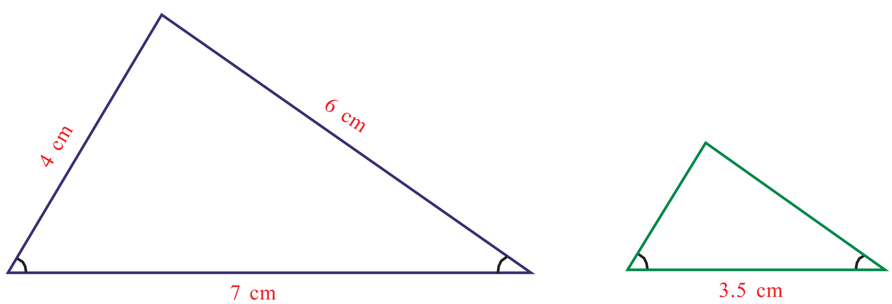
Angles and sides

Don't you know how to draw a triangle of specified sides? Draw a triangle of sides 4, 6 and 7 centimetres.

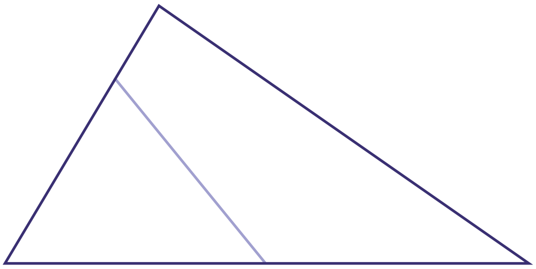


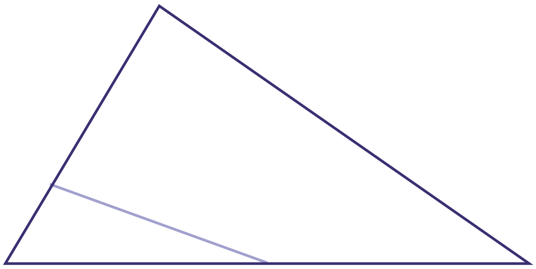
Now can you draw this a bit smaller? The longest side need only be half its length now; that is 3.5 centimetres. And another thing is that the angles must be the same. How would you draw it?

Measure all angles. Draw a line 3.5 centimetres long and on each side of it, draw angles equal to the base angles of the larger triangle.



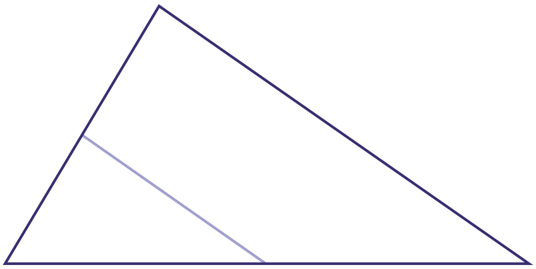
Is there any easier way to do it, without measuring the angles? Let's see if we can reduce the longest side by half and draw within the large triangle itself. If we draw any line through the midpoint of the base, we would get a triangle with one side 3.5 centimetres and one angle the same as that of the large triangle.





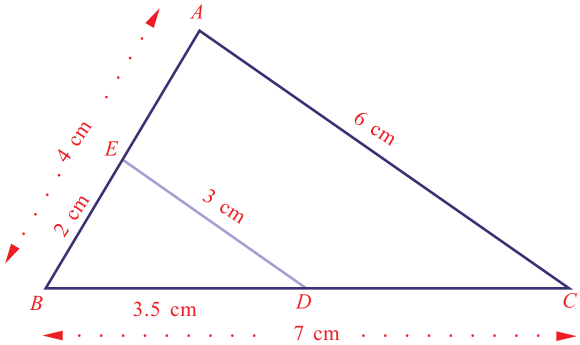
But in the first picture, the angle on the right in the small triangle is a bit too large and the angle on top is a bit too small; in the second picture, it is the other way round. How do we make the angles correct?

How about drawing parallel to the right side?



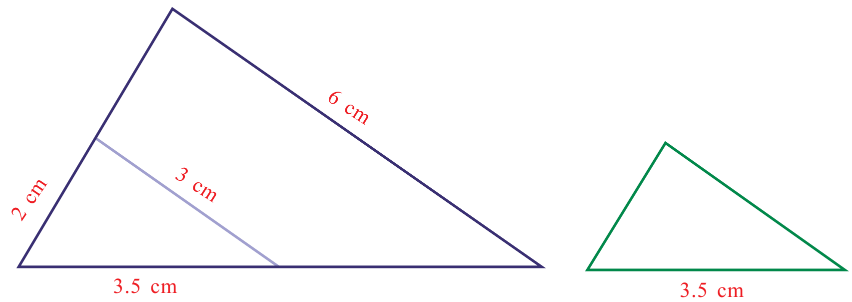
Now the angles are okay, right? (How?)

We note another thing here. The line inside the triangle is drawn through the midpoint of the bottom side and parallel to the right side. So, it bisects the left side; and its length is half that of the right side (The section, **Triangle division** in the lesson, **Parallel Lines**).

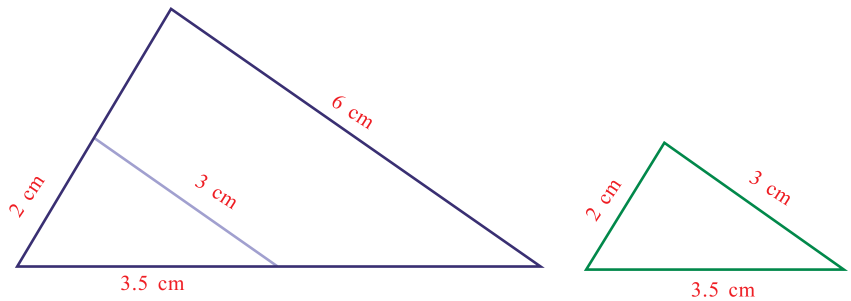


Thus when we drew without changing the angles and only the longest side halved, the other two sides were also halved.

Is this true for the first triangle drawn by measuring angles?



The small triangle on the left and the triangle on the right have one equal side. And the way we drew them, the angles on both ends of this line are also the same. So, the other two sides of these triangles are also equal. (The lesson, **Equal Triangles** in the Class 8 textbook).



This is true, whatever be the lengths of the first triangle we started with. So, if we halve one side of a triangle without changing angles, the other sides would also be halved.

There are some questions at this point:

- Instead of making it half, if we shrink or stretch by any other number, would this be true?
- Instead of the longest side, if we take the shortest side or the medium one, would this be true?

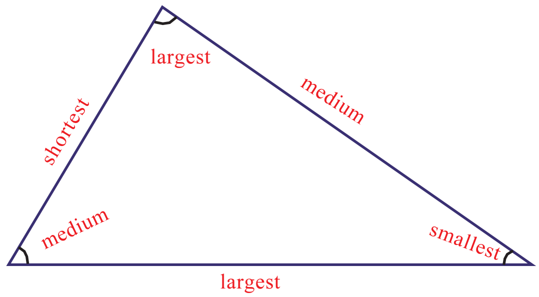
We can combine these into one question, by using terms of ratios.

In two triangles of the same angles, is the ratio of the shortest, medium and longest sides the same?

In this, instead of distinguishing the sides as the longest,

Draw ΔABC in GeoGebra and mark all its angles. Make a Slider d with Min = 0. Use **Segment with Given Length** to draw DE of length d times AB. For this, the length of DE can by given as dAB. Next draw ΔABC with $\angle D = \angle A$ and $\angle E = \angle B$. For this, choose **Angle with Given Size** and click on E and D. In the window, give the size of the angle as α (the size of $\angle A$). Similarly click on D, E and give the size of angle as β size of (the size of $\angle B$) with **counter clockwise** selected. Join DE′, ED′ and mark their point of intersection as F. Mark the sides of both triangles. Are the ratios the same? Change the angles using sliders and see.

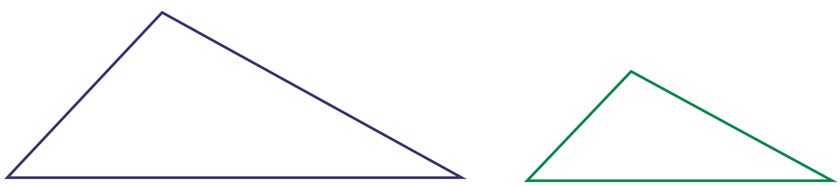
medium and shortest, there is another way. The size of the sides depends on the angles opposite them, right?



So our question can be put this way:

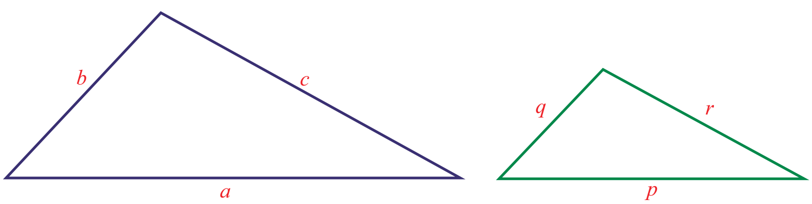
> In triangles of the same angles, are all pairs of sides opposite to equal angles in the same ratio?

To check this, let's look at two triangles with the same angles:

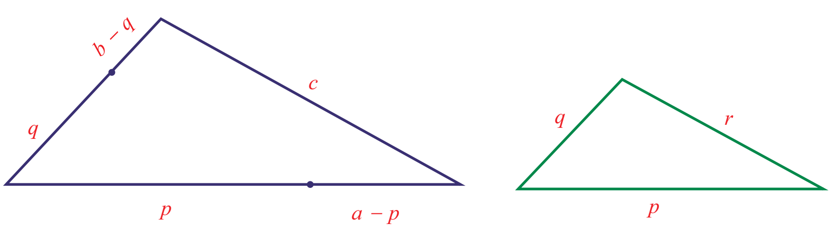


In these triangles, the left, right and top angles are the same. We have to check whether the right, left and bottom sides are in the same ratio.

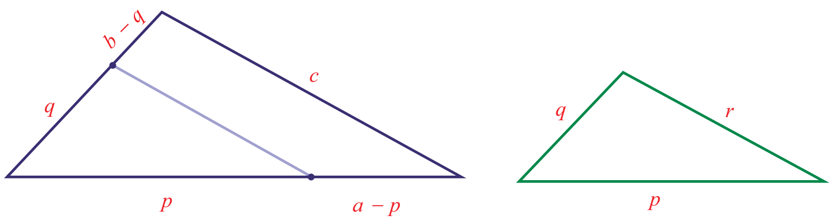
Let's take the lengths of the sides of the large triangle as a, b, c and those of the small triangle as p, q, r as shown in the pictures below:



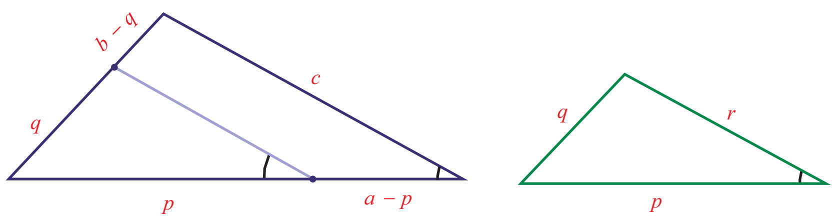
First let's compare the bottom and left sides. For this, we mark the shorter lengths along the longer ones:



And join these positions:



Now in the small triangles inside and outside the large triangle the bottom and left sides equal; so is the angle between them. So, their other angles are also equal (The lesson, **Equal Triangles** in Class 8). The angles of the outer triangle are those of the large triangle. Thus the three angles marked in the picture below are all equal:



Since the right side of the large triangle and the line inside make the same angle with the bottom side, they are parallel. So, the line inside divides the other two sides in the same ratio: That is,

$$\frac{a-p}{p} = \frac{b-q}{q}$$

Simplifying this, we get

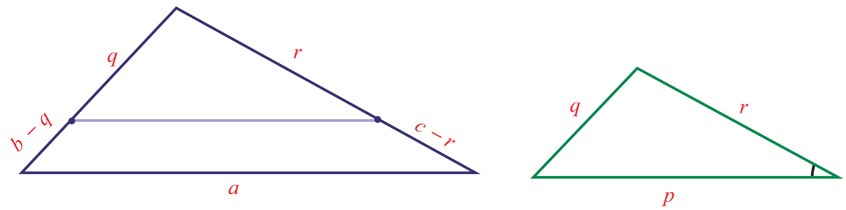
$$\frac{a}{p} - 1 = \frac{b}{q} - 1$$

And this gives

$$\frac{a}{p} = \frac{b}{q}$$

Similarly, by marking the left and right sides of the outer triangle along the left and right sides of the large triangle, we get

$$\frac{b-q}{q} = \frac{c-r}{r}$$



And from it,

$$\frac{b}{q} = \frac{c}{r}$$

Combining the two equations, we write

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$$

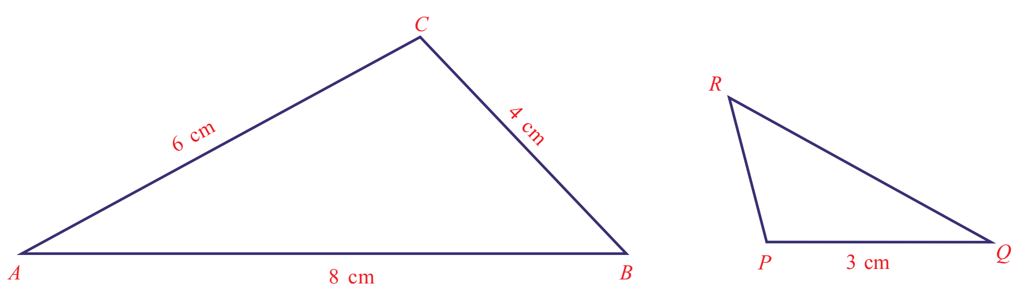
Since there is nothing special about the sides or angles of the triangles we have taken, this is true for all such triangles.

In triangles of the same angles, pairs of sides opposite equal angles are in the same ratio.

This can be put another way: In two triangles of the same angles, whatever multiple of the shortest side of one is the shortest side of the other, the medium and longest sides are also the same multiples. In general, enlarging or shrinking a quantity by multiplying by a fixed number is called *scaling* and the number is called the *scale factor*. Thus our result on triangles can be put this way:

In triangles of the same angles, the sides are scaled by the same factor.

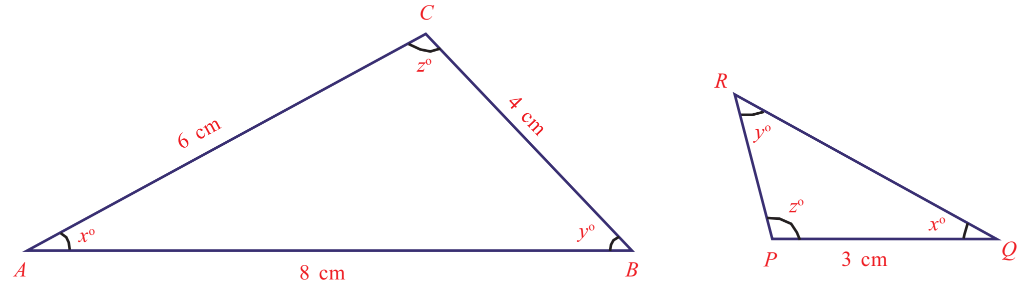
Let's look at some problems based on this. See these two triangles:



$$\angle P = \angle C \quad \angle Q = \angle A \quad \angle R = \angle B$$

How do we calculate the lengths of the other two sides of the small triangle?

First let's take the angles as x°, y°, z° and mark equal angles, as given, in the picture:



Then we write the pairs opposite equal angles:

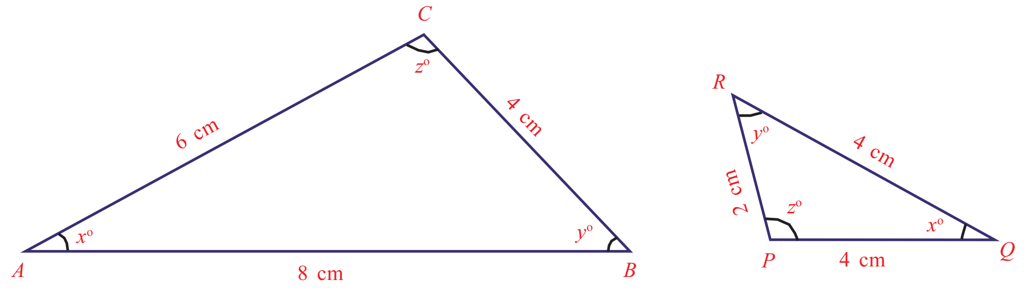
x	BC	PR
y	AC	PQ
z	AB	QR

In this, we know the lengths of all sides of the large triangle and the length of one side of the small triangle:

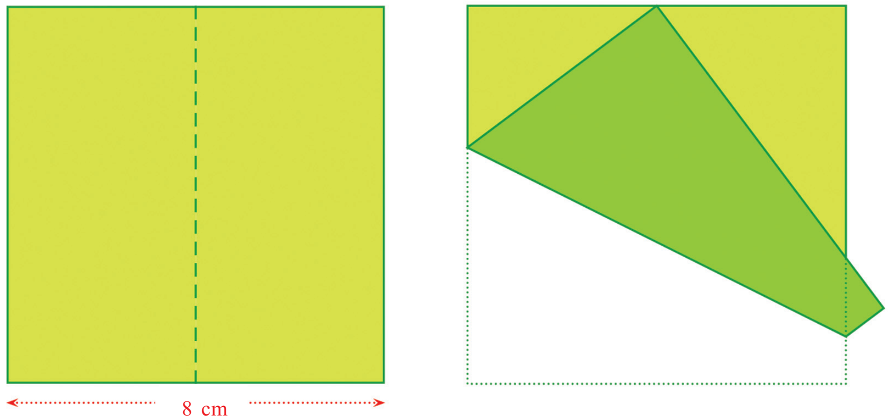
x	BC	$= 4$	PR
y	AC	$= 6$	$PQ = 3$
z	AB	$= 8$	QR

We see that for the sides opposite the y° angle, the smaller is half the larger. So the sides opposite the other angles must also be related in the same manner:

x	BC	$= 4$	$PR = 2$
y	AC	$= 6$	$PQ = 3$
z	AB	$= 8$	$QR = 4$



Another problem: one bottom corner of a square sheet of paper, of sides 8 centimetres, is folded up to the mid point of the top side.

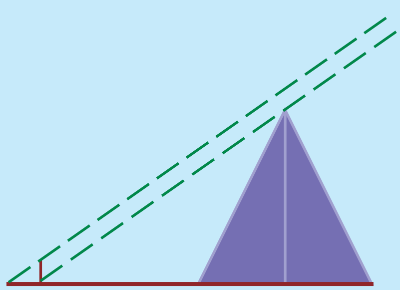


Now we get two right triangles on top. Let's compute the length of their sides.

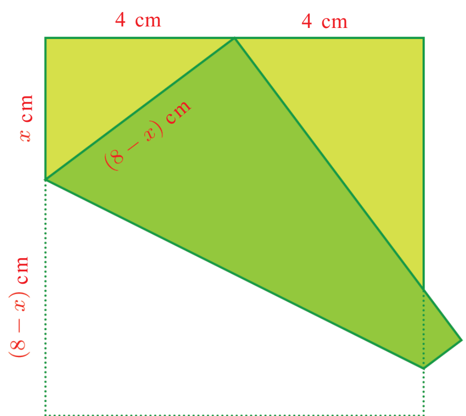
An old shadow math

Haven't you heard the story of how the Greek mathematician Thales determined the distance to a ship at sea, using the idea of equality of triangles?

There's another tale about Thales. The king of Egypt is supposed to have asked Thales to compute the height of a pyramid. Thales' method is recorded like this. "Planting his staff at the end of the shadow of the pyramid, he showed that in the two triangles made by sun rays, the ratio of the shadows is the same as the ratio of the staff and the pyramid".



Taking the vertical side of the left triangle as x centimetres, we can mark the lengths of sides as shown below:



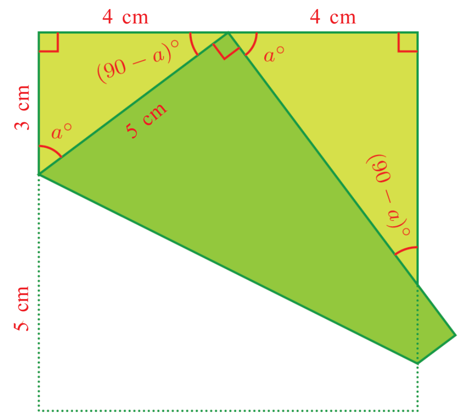
So, the relations between the sides of this right triangles can be written using Pythagoras Theorem:

$$(8-x)^2 - x^2 = 4^2$$

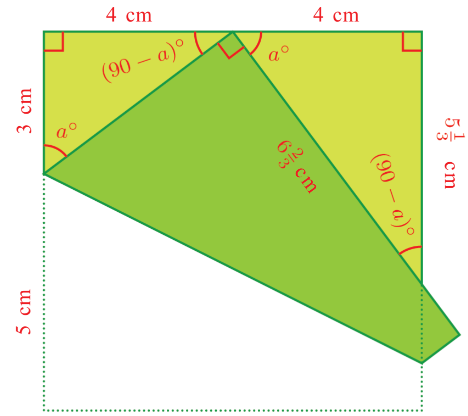
Simplifying this, we get

$$8\,(8-2x) = 16$$

from which we get $x = 3$, so the sides of this triangle are 3, 4 and 5 centimetres.



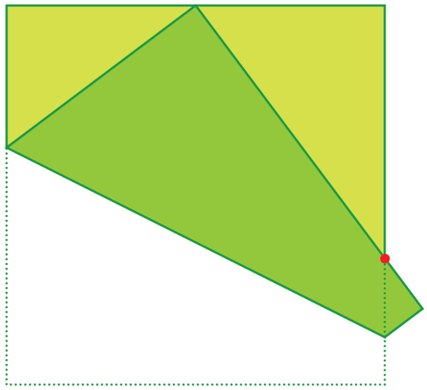
To compute the sides of the other triangle, let's look at the angles of the triangles. Taking the lower angle of the left triangle as a°, the other angles can be written as shown in the picture.

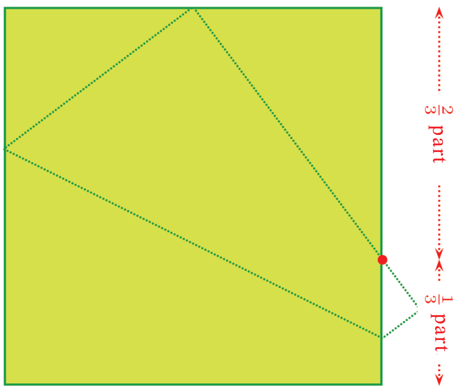


Since the triangles have the same angles, the sides opposite equal angles must be scaled by the same factor. The side opposite $(90 - a)^\circ$ angle is 3 centimetres in the triangle on the left and 4 centimetres in the triangle in the right. So the other perpendicular side and the hypotenuse of the triangle on the right must be $\frac{4}{3}$ times these sides of the triangle on the left. Thus the lengths of its sides are as shown in the picture.

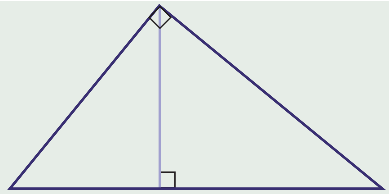
Would the ratios of the sides of the two right triangles got by folding a square piece of paper of any size be 3 : 4 : 5?

There is another thing: If we mark the point where the folded side intersects the right side and open up the sheet, we would get the $\frac{1}{3}$ length of the right side.





(1) The perpendicular from the square corner of a right triangle cuts the opposite side into two parts of 2 and 3 centimetres length.



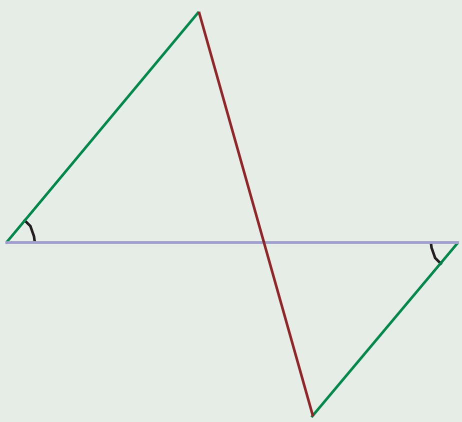
i) Prove that the two small right triangles cut by the perpendicular have the same angles.

ii) Taking the length of the perpendicular as h, prove that $\frac{h}{2} = \frac{3}{h}$.

iii) Calculate the perpendicular sides of the large triangle.

iv) Prove that if the perpendicular from the square corner of a right triangle divides the opposite side into parts of lengths a and b and if the length of the perpendicular is h, then $h^2 = ab$.

(2) At two ends of a horizontal line, angles of equal size are drawn, and some points on the slanted lines are joined:

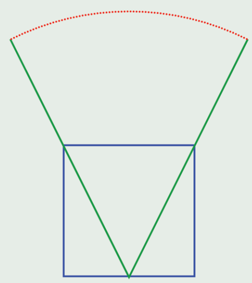
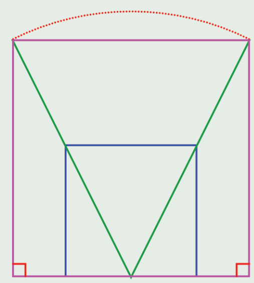


i) Prove that the parts of the horizontal line and parts of the slanted line are in the same ratio.

ii) Prove that the two slanted lines at the ends of the horizontal line are also in the same ratio.

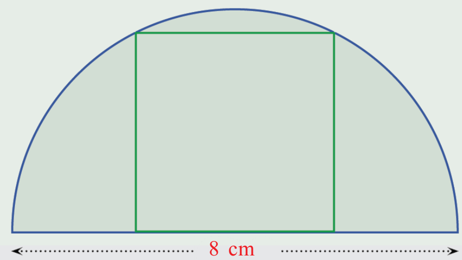
iii) Explain how a line of length 6 centimetres can be divided in the ratio 3 : 4 using this.

(3) The midpoint of the bottom side of a square is joined to the ends of the top side and extended by the same length. The ends of these lines are joined and perpendiculars are drawn from these points to the bottom side of the square extended:

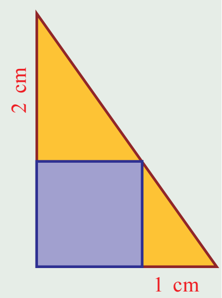



i) Prove that the quadrilateral obtained thus is also a square.

ii) Explain how we can draw a square with two corners on the diameter and the others on a semicircle, as shown in the picture.



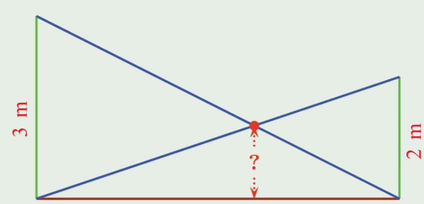
(4) The picture shows a square drawn sharing one corner with a right triangle and the other three corners on the sides of this triangle.



i) Calculate the length of a side of the square.

ii) What is the length of a side of the square drawn like this within a triangle of sides 3, 4 and 5 centimetres?

(5) Two poles of heights 3 metres and 2 metres are erected upright on the ground and ropes are stretched from the top of each to the foot of the other.



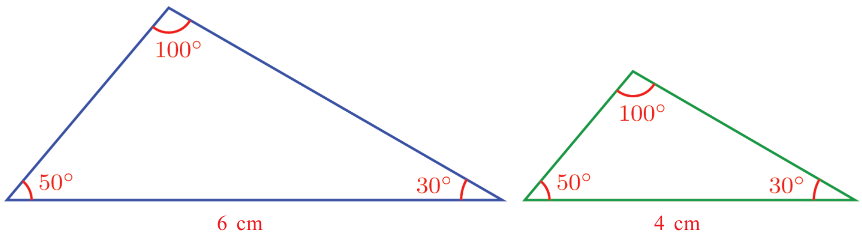
i) At what height above the ground do the ropes cross each other?

ii) Prove that this height would be the same, whatever be the distance between the poles.

iii) Taking the heights of the poles as a and b and height above the ground of the point where the ropes cross each other as h, find the relation between a, b and h.

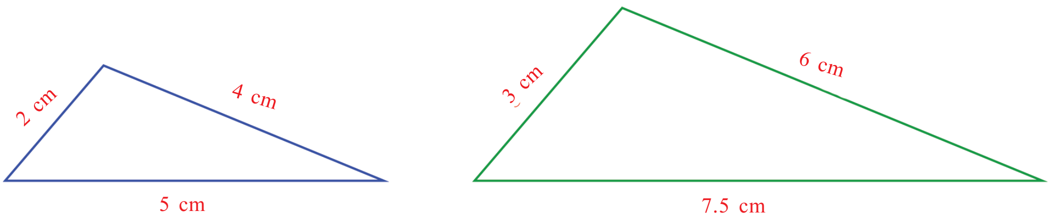
Sides and angles

We have seen that if two triangles have the same angles, then their sides are scaled by the same factor. For example, look at these triangles:



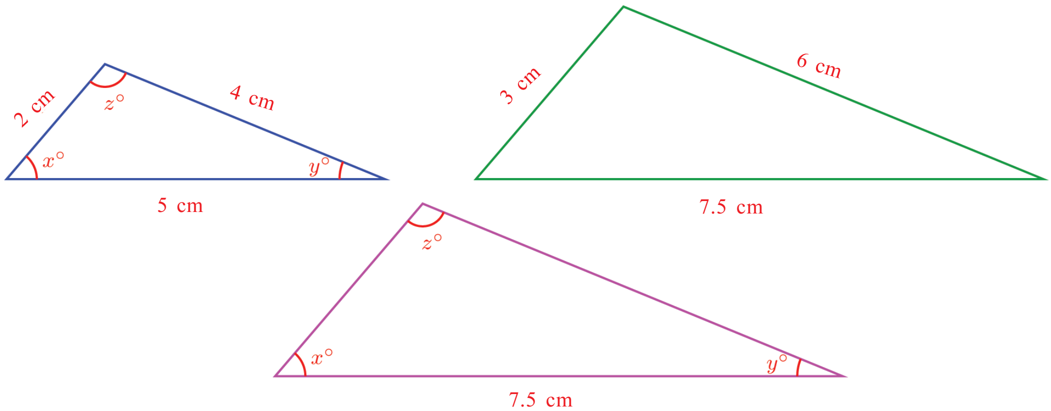
They have the same angles. The longest side of the small triangle is $\frac{4}{6} = \frac{2}{3}$ of the longest side of the large triangle. So, the shortest side of the small triangle is also $\frac{2}{3}$ of the shortest side of large triangle; the same is true for the medium sides also.

Now we can ask in reverse: if the sides of a triangle are scaled by the same factor, would the angles remain the same?

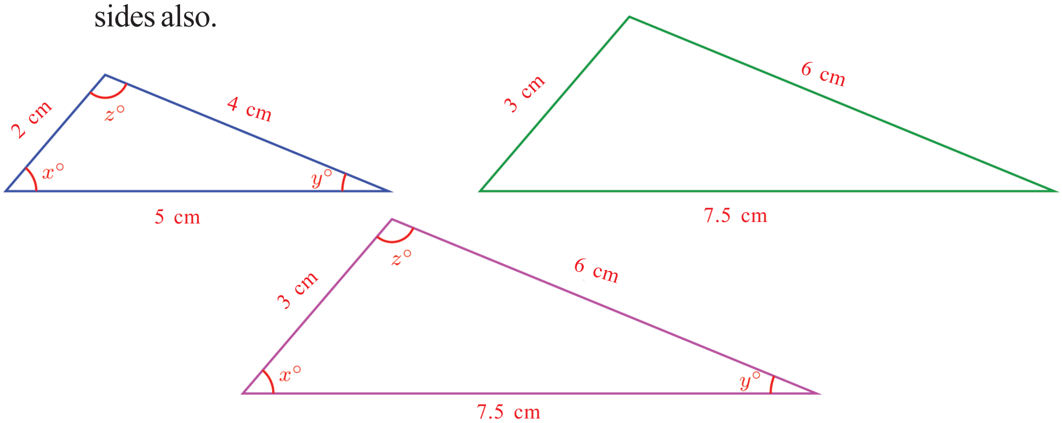


In the picture above, the sides of the larger triangle are one and a half times the sides of the smaller triangle. Do they have the same angles?

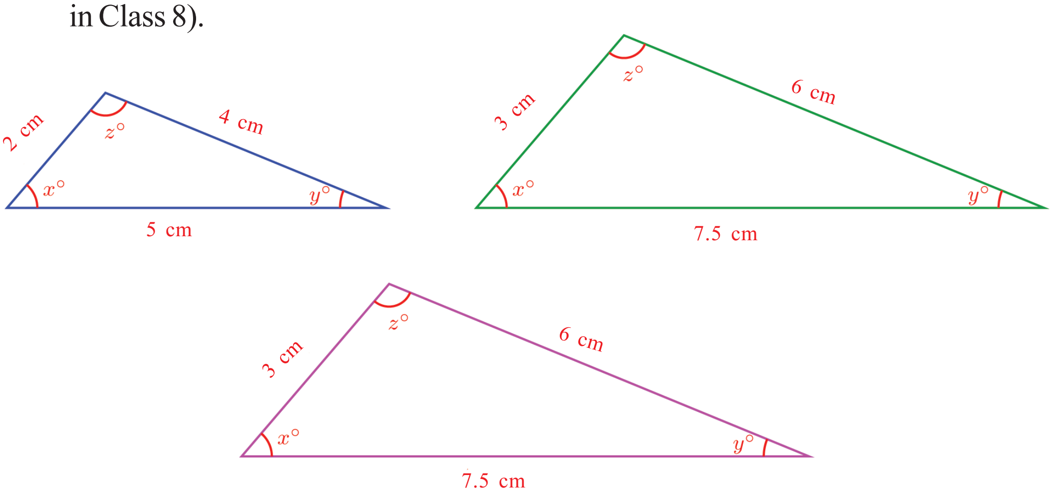
To check this, we draw a third triangle. Its longest side must equal to the longest side of the large triangle; the angles at its ends have to be equal to the angles at the ends of the longest side of the small triangle:



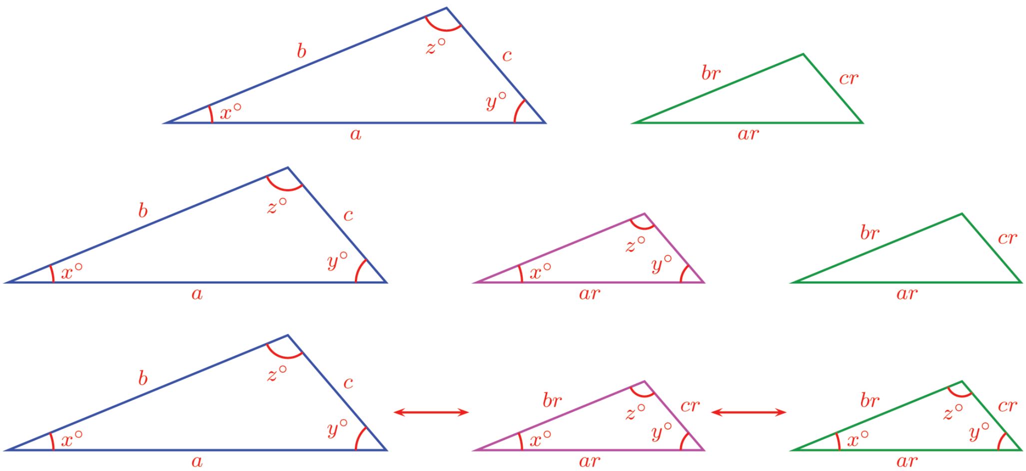
Since the angles of this new triangle are the same as the angles of the small triangle, the sides of these two triangles must be scaled by the same factor. Since it is one and a half for the longest side, it must be the same for other sides also.



Thus the sides of this new triangle are equal to the sides of the old large triangle. So, their angles must also be the same (The lesson, **Equal Triangles** in Class 8).

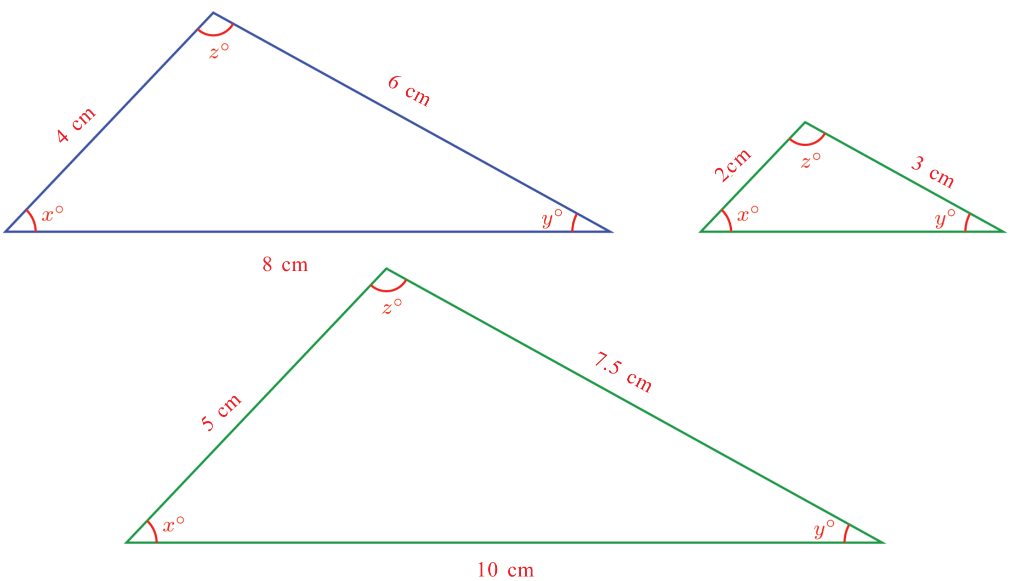


Thus we find that the angles of the first two triangles are the same. For any two triangles with sides scaled by the same factor, we can see that the angles are the same, with the help of an intermediary triangle like this:



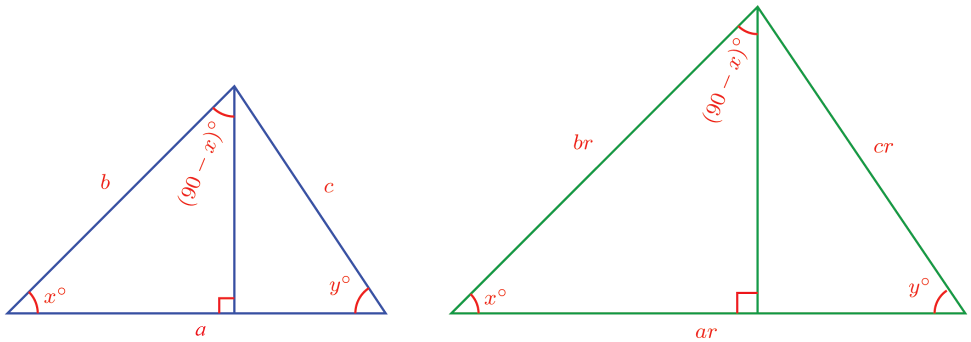
In triangles of sides scaled by the same factor, the angles are the same.

So to transform a triangle into a smaller or larger one without changing angles, we need not measure the angles, we need only scale the sides by the same factor.

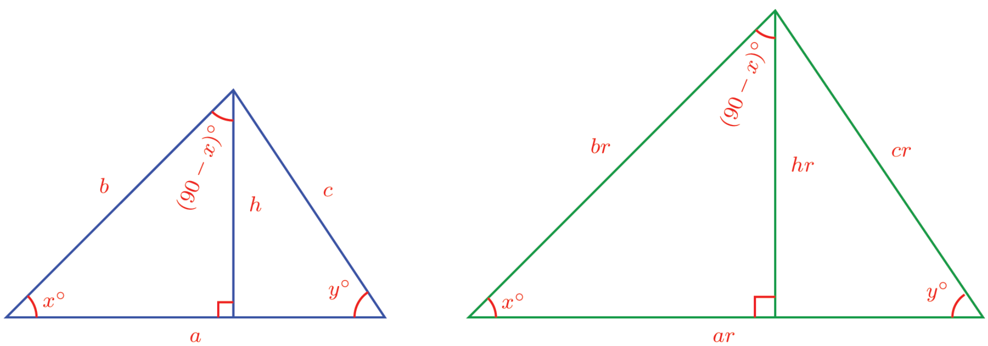


Let's look at a problem based on this. We can easily see that if the sides of a triangle are scaled by the same factor, then their perimeters are also scaled by the same factor. (Try it!)

How are the areas related? To see this, let's draw two such triangles. By what we have seen just now, they have the same angles. To compare the areas, let's draw perpendiculars from two vertices with the same angles.



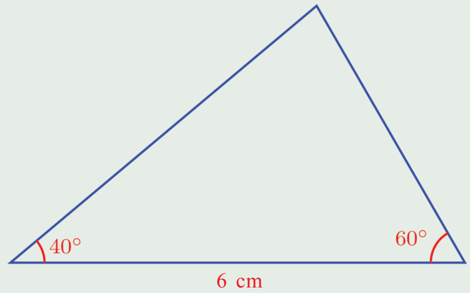
Look only at the right triangles on the left of each. Both have angles x°, 90° and $(90 - x)^\circ$. So their sides are scaled by the same factor. The hypotenuse of the blue right triangle is b and that of the green right triangle is br. So if we take the perpendicular in the blue triangle as h, the perpendicular in the green triangle is hr.



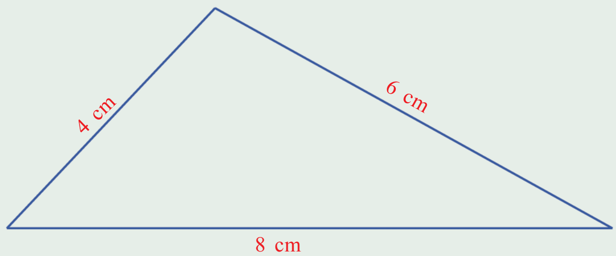
Now we can compute the areas of both the whole triangles. The area of the blue triangle is $\frac{1}{2}ah$ and the area of the green triangle is $\frac{1}{2}ahr^2$.

Thus the scale factor of areas is the square of the scale factor of the sides.

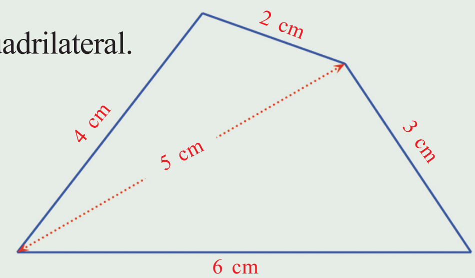
(1) Draw a triangle of angles the same as those of the triangle shown here and sides scaled by $\frac{3}{4}$.



(2) Draw a triangle of angles the same as those of the triangle shown and sides scaled by $1\frac{1}{4}$.



(3) See this picture of a quadrilateral.



i) Draw a quadrilateral with angles the same as those of this one and sides scaled by $1\frac{1}{2}$.

ii) Draw a quadrilateral with angles different from those of this and sides scaled by $1\frac{1}{2}$.

Triangle speciality

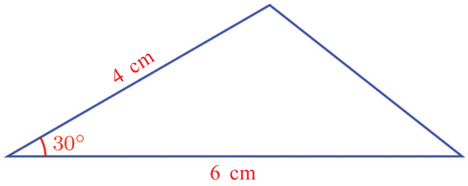
If the angles of a triangle are equal to the angles of another, then their sides are in the same ratio. And conversely, if the sides of two triangles are in the same ratio, then their angles are equal. Among polygons, only triangles have this property.

The third way

If we know one side of a triangle and the angles at its ends, the first part of our discussion shows how we can scale it without altering its angles. Scale the known side and draw the same angles at its ends. The other sides would be scaled by the same factor.

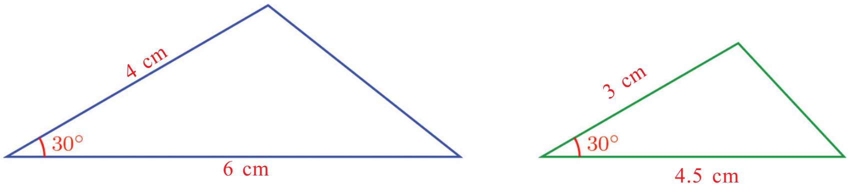
If what we know are the lengths of the three sides, then the second part of the discussion shows how it can be scaled. Just scale all sides by the same factor; the angles would remain the same.

Now suppose that we know are the sides of the triangle to be scaled and the angle between them. For example, see the triangle on the right.



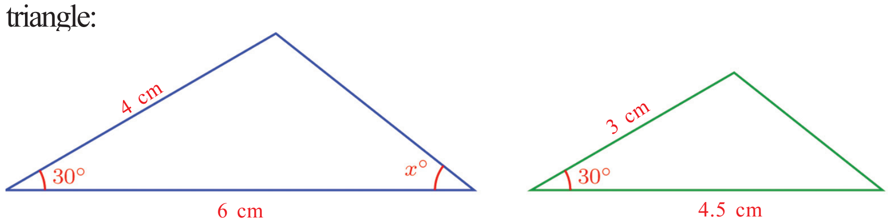
We want to scale it by $\frac{3}{4}$.

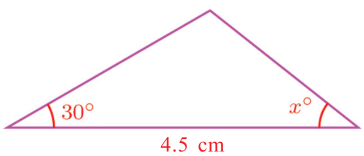
We can draw a triangle of sides $\frac{3}{4}$ of 6 and 4 centimetres and the angle between them as 30°.



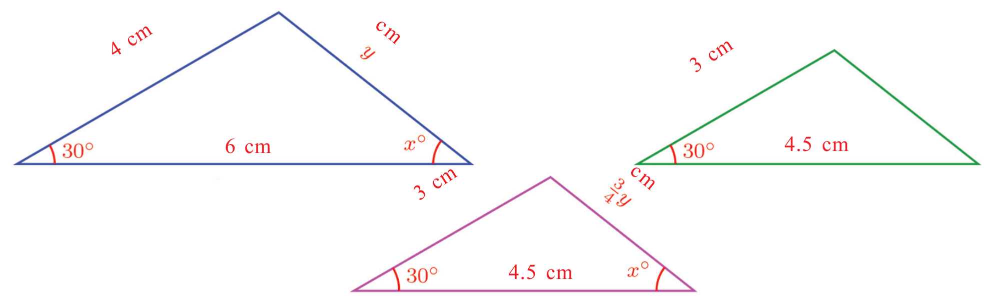
But we don't know whether the third side also is $\frac{3}{4}$ the third side of the first triangle.

To check this, we draw an intermediary triangle as before with bottom side 4.5 centimetres, angles on its ends equal to the bottom angles of the large triangle:

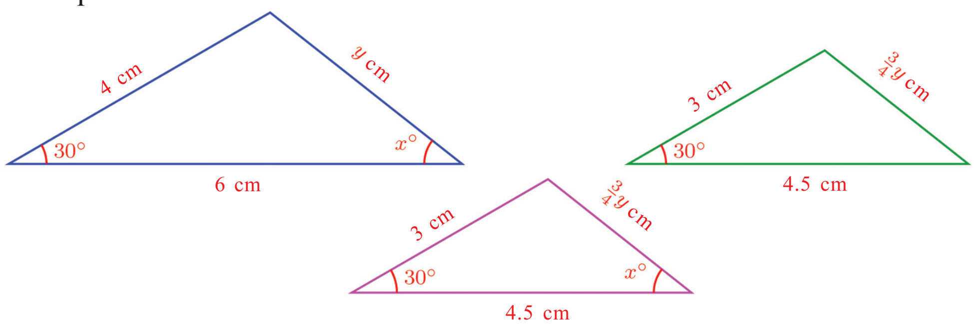




Since the angles in the new triangle are equal to those of the large triangle, the sides of these triangles must be scaled by the same factor. The bottom side of the new triangle is $\frac{3}{4}$ the bottom side of the large triangle. So, the other sides also scaled by $\frac{3}{4}$. If the length of the unknown side of the large triangle is taken as y centimetres, the sides of the new triangle can be written as below:

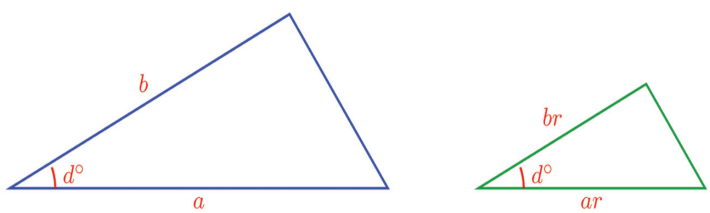


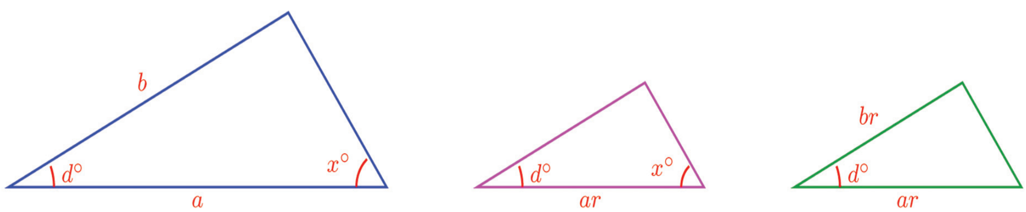
Now let's compare the new triangle and the old small triangle. In these, the two sides and the angle between them are equal. So, the third sides are also equal.

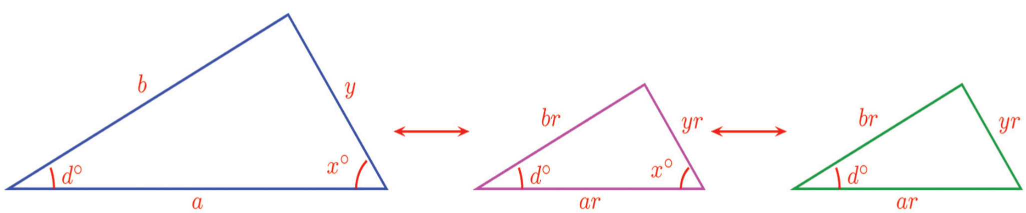


Thus we see that the third side of the first small triangle is also $\frac{3}{4}$ the third side of the large triangle.

We can draw an intermediary triangle like this, whatever be the sides and angle of the first triangle.

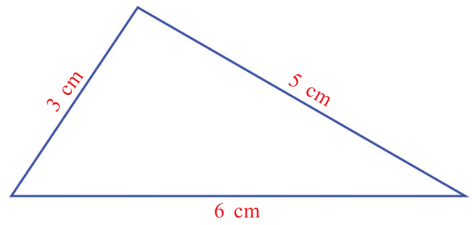




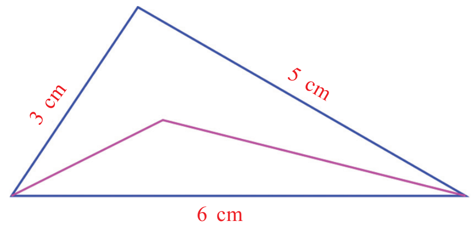


In triangles of two sides scaled by the same factor and the angle between them the same, the third sides are also scaled by the same factor.

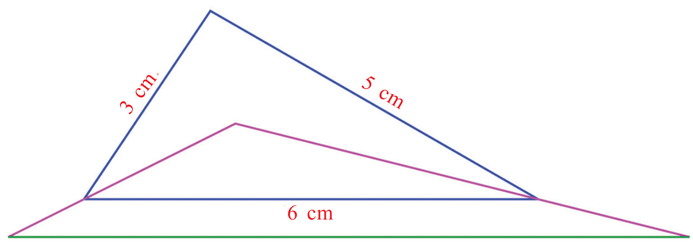
Using this, we can scale a triangle without measuring sides or angles. For example, draw a triangle like this.



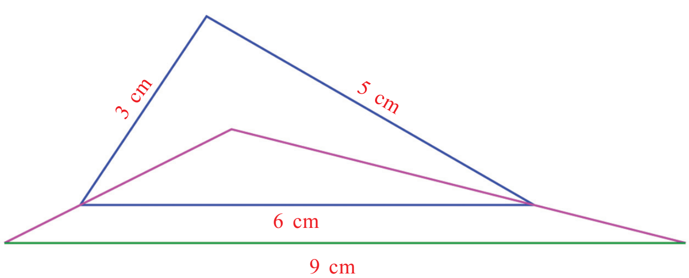
Mark some point inside the triangle and join it to the ends of the bottom side.



Extend each of these lines by its half and join the ends.

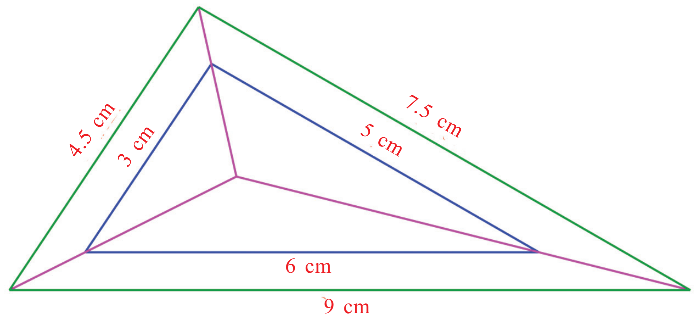


Now we have a new triangle and a small one inside it. The left and right sides of the large triangle are one and a half times those of the small triangle; and the angle between these is the same for both triangles. So, the third side of the large triangle also is one and a half times the third side of the small triangle.



We can scale triangles in GeoGebra like this. Draw ΔABC and mark a point D inside or outside the triangle. Draw lines from D to A, B,C using **Ray From a Point**. Make a slider g with **Min = 0** and draw circles centred at D with radii g ∗ AD, g ∗ BD, g ∗ CD and mark the points where these lines meet AD, BD, CD as E, F, G. Hide the circles and draw the triangle. Move the slider and see what happens. What do you see when g = 1?, When g = 0.5? g = 2? Change the position of D and see what you get.

Join the point inside the triangle and the other two vertices and extend is before. What do we get?



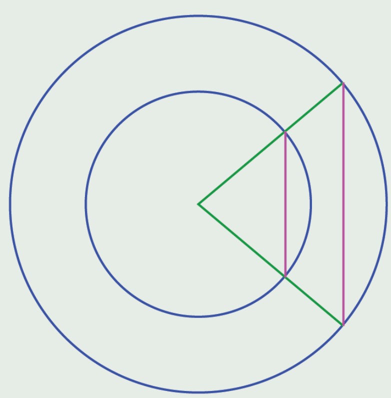
To draw similar triangles in GeoGebra. We can use **Dilate from Point**. Make a slider a with Min = 0. Draw a triangle and mark a point inside it or outside it. Choose **Dilate from Point** and click on the point ands triangle. In the new window give a as **Scale Factor**. We get a triangle similar to the first one. Move the slider to change a and see what happens. The position of D may also be changed. We can draw similar shapes of any shape like this.

All sides of the large triangle are one and a half times that of the original triangle, right?

Two triangles of sides scaled by the same factor are called *similar*. By the general principles we have seen, for two triangles to be similar, they have to be related in one of the following ways:

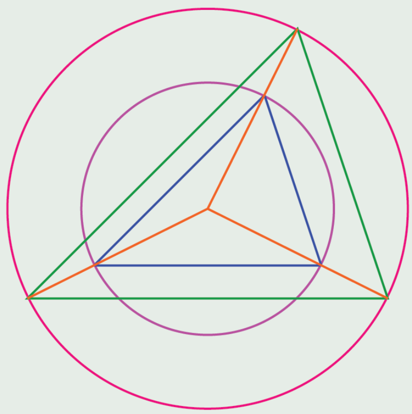
- Having the same angles.
- Having sides scaled by the same factor .
- Having two sides scaled by the same factor and the angles between them equal.

(1) The picture shows two circles with the same centre and two triangles formed by joining the centre to the points of intersection of the circles with two radii of the larger circle:



Prove that these triangles are similar.

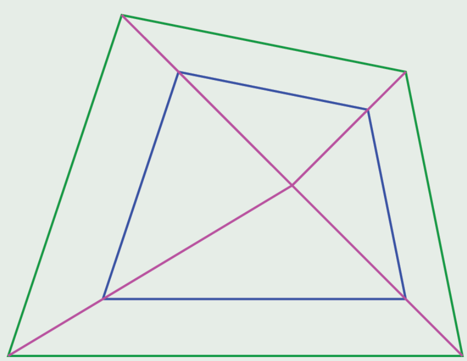
(2) The lines joining the circumcentre of a triangle to the vertices are extended to meet another circle with the same centre, and these points are joined to make another triangle.



i) Prove that the two triangles are similar.

ii) Prove that the scale factor of the sides of the triangle is the scale factor of the radii of the circles.

(3) A point inside a quadrilateral is joined to its vertices and the lines are extended by the same scale factor. Their ends are joined to make another quadrilateral.



i) Prove that the sides of the two quadrilaterals are scaled by the same factor.

ii) Prove that the angles of the two quadrilaterals are the same.

Project

In similar triangles, how are the angle bisectors, medians and the circumradii related?

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Describing that in triangles of the same angles, sides are scaled by the same factor.			
• Scaling a triangle with one side and angles at its ends known.			
• Recognising that triangles of sides scaled by the same factor have the same angles.			
• Scaling a triangle with three sides known.			
• Explaining that in triangles of two sides scaled by the same factor and the angle between them equal, the third sides are also scaled by the same factor.			
• Scaling a triangle without measuring sides or angles.			
• Scaling any polygon without measuring sides or angles.			

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